

Solutions of the Nonlinear Schrödinger Equation with Prescribed Asymptotics at Infinity

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Abstract

We prove local existence and uniqueness of solutions for the one-dimensional nonlinear Schrödinger (NLS) equations $iu_t + u_{xx} \pm |u|^2 u = 0$ in classes of smooth functions that admit an asymptotic expansion at infinity in decreasing powers of x . We show that an asymptotic solution differs from a genuine solution by a Schwartz class function which solves a generalized version of the NLS equation. The latter equation is solved by discretization methods. The proofs closely follow previous work done by the author and others on the Korteweg-De Vries (KdV) equation and the modified KdV equations.

1 Introduction

In this article we consider the one-dimensional focusing and defocusing nonlinear Schrödinger equations

$$\begin{cases} iw_t + w_{xx} + \mu|w|^2w &= 0 \\ w|_{t=0} &= w_0(x) \end{cases} \quad (1.1)$$

where $\mu = \pm 1$ and the initial data w_0 has prescribed asymptotic expansions at $+\infty$ and/or $-\infty$. Specifically we shall construct local (in time) solutions to (1.1) that lie in the spaces $S^\beta(\mathbb{R} \times I \rightarrow \mathbb{C})$

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which are complex analogues to those spaces defined in [13] by T. Kappeler, P. Perry, M. Shubin, and P. Topalov and used also by the author in [9]. These spaces are defined as follows:

Let $I \subset \mathbb{R}$ be an interval and $\beta \in \mathbb{R}$ be given. Denote by $S^\beta(\mathbb{R} \times I \rightarrow \mathbb{C})$ (or simply $S^\beta(\mathbb{R} \times I)$) the linear space of $C^\infty(\mathbb{R} \times I \rightarrow \mathbb{C})$ functions having asymptotic expansions at $\pm\infty$ given by $f(x, t) \sim \sum_{k=0}^{\infty} (a_k^+(t) + ib_k^+(t)) x^{\beta_k}$ as $x \rightarrow +\infty$ and $f(x, t) \sim \sum_{k=0}^{\infty} (a_k^-(t) + ib_k^-(t)) x^{\beta_k}$ as $x \rightarrow -\infty$ where $a_k^\pm, b_k^\pm \in C^\infty(I \rightarrow \mathbb{R})$ and $\beta = \beta_0 > \beta_1 > \dots$ with $\lim_{k \rightarrow \infty} \beta_k = -\infty$. By definition, the asymptotic relation \sim means that for every compact interval $J \subset I$, and integers $N, i, j \geq 0$ there exists $C_{J, N, i, j} > 0$ such that for any $\pm x \geq 1$ and $t \in J$ we have

$$\left| \partial_t^i \partial_x^j \left(f(x, t) - \sum_{k=0}^N (a_k^\pm(t) + ib_k^\pm(t)) (\pm x)^{\beta_k} \right) \right| \leq C_{J, N, i, j} |x|^{\beta_{N+1}-j}$$

We denote by $S^{-\infty}(\mathbb{R} \times I \rightarrow \mathbb{C})$ (or simply $S^{-\infty}(\mathbb{R} \times I)$) the space of $C^\infty(\mathbb{R} \times I \rightarrow \mathbb{C})$ functions having asymptotic expansions at $\pm\infty$ which are identically zero. Analogously, we define the spaces $S^\beta(\mathbb{R} \rightarrow \mathbb{C})$ and $S^{-\infty}(\mathbb{R} \rightarrow \mathbb{C})$ as the space of functions $f(x) \in C^\infty(\mathbb{R} \rightarrow \mathbb{C})$ having such asymptotic expansions where the coefficients a_k^\pm, b_k^\pm are constants independent of t . We shall construct solutions $w(x, t) \in S^\beta(\mathbb{R} \times I)$ for (1.1) with initial data $w_0 \in S^\beta(\mathbb{R})$ when $\beta \leq 0$.

If $w(x, t) \in S^\beta(\mathbb{R} \times I)$ is a solution for (1.1) then one expects its asymptotic expansions $\sum_{k=0}^{\infty} (a_k^\pm(t) + ib_k^\pm(t)) (\pm x)^{\beta_k}$, although not generally convergent, to give formal solutions (see lemma 5.2). We define a pair of formal power series $\sum_{k=0}^{\infty} (a_k^\pm(t) + ib_k^\pm(t)) x^{\beta_k}$ to be a formal solution to (1.1) if $\sum_{k=0}^{\infty} (a_k^+(t) + ib_k^+(t)) x^{\beta_k}$ and $\sum_{k=0}^{\infty} (a_k^-(t) + ib_k^-(t)) (-x)^{\beta_k}$ satisfy (1.1) for all $t \in I$ when x is taken as a formal variable and differentiation in x is carried out in the ordinary way.

When $\beta > 0$ one can easily see that there are no such formal solutions to (1.1) and hence no solutions in $S^\beta(\mathbb{R} \times I)$. Indeed, if $\sum_{k=0}^{\infty} (a_k^+(t) + ib_k^+(t)) x^{\beta_k}$ satisfies (1.1) formally where $\beta > 0$ and $a_0^+ + ib_0^+ \neq 0$ then

$$\begin{aligned} i \sum_{j=0}^{\infty} (\dot{a}_j^+(t) + ib_j^+(t)) x^{\beta_j} + \left(\sum_{j=0}^{\infty} (a_j^+(t) + ib_j^+(t)) \cdot \beta_j \cdot (\beta_j - 1) x^{\beta_j - 2} \right) = \\ \mu \left(\sum_{j=0}^{\infty} (a_j^+(t) + ib_j^+(t)) x^{\beta_j} \right)^2 \cdot \left(\sum_{j=0}^{\infty} (a_j^+(t) - ib_j^+(t)) x^{\beta_j} \right) \end{aligned}$$

The largest exponent on the left side is β_0 and the largest exponent on the right side is $3\beta_0$ which is larger than β_0 . Therefore by equating the coefficients of $x^{3\beta_0}$ one deduces that $0 = \mu(a_0^+ + ib_0^+)^2(a_0^+ - ib_0^+)$ which implies that $a_0^+ = 0 = b_0^+$, a contradiction. On the other hand when $\beta \leq 0$ there do exist such formal solutions defined for $t \in \mathbb{R}$ (see lemma A.2) and therefore

one can hope to find solutions in $S^\beta(\mathbb{R} \times I)$ for some interval I .

For an arbitrarily chosen pair of such formal power series $\sum_{k=0}^{\infty} (a_k^\pm(t) + ib_k^\pm(t)) (\pm x)^{\beta_k}$ defined for $t \in I$ there exists a function $f(x, t) \in C^\infty(\mathbb{R} \times I \rightarrow \mathbb{C})$ asymptotic to the pair (see for example [17] proposition 3.5). The function f is not unique but if $\sum_{k=0}^{\infty} (a_k^\pm(t) + ib_k^\pm(t)) (\pm x)^{\beta_k}$ is a formal solution then any such f will be an asymptotic solution for (1.1) (see lemma A.2). By definition an asymptotic solution is a function $f \in S^\beta(\mathbb{R} \times I \rightarrow \mathbb{C})$ such that

$$\begin{cases} if_t + f_{xx} + \mu|f|^2 f & \in S^{-\infty}(\mathbb{R} \times I \rightarrow \mathbb{C}) \\ f|_{t=0} - w_0 & \in S^{-\infty}(\mathbb{R} \rightarrow \mathbb{C}) \end{cases}$$

Given an asymptotic solution $f(x, t) \in S^\beta(\mathbb{R} \times I)$ for (1.1) one can attempt to construct a genuine solution $w(x, t) \in S^\beta(\mathbb{R} \times I)$ to (1.1) by constructing $u(x, t) \in S^{-\infty}(\mathbb{R} \times I)$ such that $w := f + u$ is a genuine solution of (1.1). If w satisfies (1.1) then u must satisfy

$$\begin{cases} iu_t + u_{xx} + \mu(u^2\bar{u} + u^2\bar{f} + f^2\bar{u} + 2uf\bar{u} + 2uf\bar{f}) + g & = 0 \\ u|_{t=0} & = u_0(x) \end{cases} \quad (1.2)$$

where $u_0 = w_0 - f(x, 0) \in S^{-\infty}(\mathbb{R} \rightarrow \mathbb{C})$ and $g \in S^{-\infty}(\mathbb{R} \times I \rightarrow \mathbb{C})$ is the result of plugging f into (1.1).

We shall prove existence of finite time solutions $u(x, t) \in S^{-\infty}(\mathbb{R} \times [0, T])$ to (1.2) by using a finite difference method as in [3, 9, 15]. Moreover, uniqueness will also be proven so that we shall show the following theorem:

Theorem 1.1 *Let $f \in C^\infty(\mathbb{R} \times [0, \infty] \rightarrow \mathbb{C})$ be a function satisfying the property that for every compact $J \subset [0, \infty]$ we have $|f^{(n)}(x, t)| = O(|x|^{-n})$ uniformly for $t \in J$ and let g be any function lying in $S^{-\infty}(\mathbb{R} \times [0, \infty] \rightarrow \mathbb{C})$. Suppose $u_0 \in S^{-\infty}(\mathbb{R} \rightarrow \mathbb{C})$. Then there exists $T > 0$ such that (1.2) has a solution $u(x, t) \in S^{-\infty}(\mathbb{R} \times [0, T] \rightarrow \mathbb{C})$. Moreover, the solution u is unique in $S^{-\infty}(\mathbb{R} \times [0, T] \rightarrow \mathbb{C})$.*

The finite-time existence and uniqueness theorem for (1.2) will enable us to prove finite-time existence and uniqueness for (1.1) in the space $S^\beta(\mathbb{R} \times [0, T])$ for $\beta \leq 0$ which can be stated as the following main theorem:

Theorem 1.2 *For any $\beta \leq 0$ and for any initial condition $w_0 \in S^\beta(\mathbb{R} \rightarrow \mathbb{C})$ there exists a $T > 0$ and a unique solution $w(x, t) \in S^\beta(\mathbb{R} \times [0, T] \rightarrow \mathbb{C})$ of the initial value problem (1.1). Moreover, if $w_0 \sim \sum_{k=0}^{\infty} (a_k^\pm + ib_k^\pm) x^{\beta_k}$ and j is the smallest index such that $a_j^+ + ib_j^+ \neq 0$ (resp. $a_j^- + ib_j^- \neq 0$) then the coefficient $a_j^+(t) + ib_j^+(t)$ (resp. $a_j^-(t) + ib_j^-(t)$) in the asymptotic expansion of the solution is a nonvanishing continuous function of t and all preceding coefficients are identically zero.*

The second statement in theorem 1.2 indicates that the asymptotic decay rate of the solution is determined throughout its time of existence by the leading exponents in the asymptotic expansion of its initial data. In particular if $\beta_0 = 0$ and $a_0^\pm \neq 0$ then the solution $w(x, t)$ for (1.1) is asymptotically constant in x .

Related Work The results and methods presented here are most closely related to those presented by the author's article on the modified Korteweg-De Vries (mKdV) equation [9] and the precursor articles on the KdV equation by Bondareva, Shubin, and Menikoff [3, 4, 15]. In [9] local existence and uniqueness is proved for the mKdV equation in spaces $S^\beta(\mathbb{R} \times I \rightarrow \mathbb{R})$ for $\beta \leq \frac{1}{2}$ and in [3, 4] global existence and uniqueness is proved for the KdV equation in $S^\beta(\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R})$ for $\beta < 1$. In [15] global existence for the KdV equation is proved in slightly different spaces of functions whose asymptotic growth is of order $|x|^\beta$ for $\beta \leq 1$. For the defocusing modified KdV equation T. Kappeler, P. Perry, M. Shubin, and P. Topalov in [13] proved global existence and uniqueness in $S^\beta(\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R})$ (as well as other spaces) for $\beta < \frac{1}{2}$. For the defocusing NLS equation (i.e. $\mu = -1$) and the closely related Gross-Pitaevskii equation there are several articles where solutions which are asymptotically constant in x are considered. We mention here articles [1, 2, 5, 6, 7, 8, 10, 11, 12, 14, 19] and some of the references therein.

In section two we introduce a discretization of (1.2), give some general lemmas, and prove finite time existence for the discrete equation. Section three contains various estimates which are necessary in order to pass from discrete solutions to smooth solutions of (1.2). In section four we show how to pass from discrete solutions to smooth solutions by using a smoothing operator I_h , introduced by Stummel in [18]. The existence statements of theorems 1.1 and 1.2 are proved in section four and the uniqueness results are proved in section five.

2 Discretization of the Generalized NLS Equation

2.1 Definitions and General Setup

We shall first split equation (1.2) into its real and imaginary components and discretize those two equations. We will also define some discrete functional spaces and give some simple properties of the definitions.

Suppose that $\beta \leq 0$, $u(x, t) = u_1(x, t) + iu_2(x, t)$, $f(x, t) = f_1(x, t) + if_2(x, t)$, and $g(x, t) = g_1(x, t) + ig_2(x, t)$ where for $l = 1, 2$ we have $u_l, g_l \in S^{-\infty}(\mathbb{R} \times I \rightarrow \mathbb{R})$ and $f_l \in S^\beta(\mathbb{R} \times I \rightarrow \mathbb{R})$.

Then we may write the real and imaginary parts of equation (1.2) as

$$\partial_t u_2 - \partial_x^2 u_1 - \mu (u_1^3 + u_1 u_2^2 + 3u_1^2 f_1 + u_2^2 f_1 + 2u_1 u_2 f_2 + 3u_1 f_1^2 + u_1 f_2^2 + 2f_1 f_2 u_2) - g_1 = 0 \quad (2.1)$$

and

$$\partial_t u_1 + \partial_x^2 u_2 + \mu (u_1^2 u_2 + u_2^3 + 2u_1 u_2 f_1 + u_1^2 f_2 + 3u_2^2 f_2 + 2f_1 f_2 u_1 + u_2 f_1^2 + 3u_2 f_2^2) + g_2 = 0 \quad (2.2)$$

Obviously solving the system (2.1), (2.2) is equivalent to solving the equation (1.2) so we shall focus our efforts on the system.

For now let us fix two mesh size numbers $0 < h, k < 1$ and let us denote $x_n := nh$ and $t_j := jk$ for each $n, j \in \mathbb{Z}$. We shall let \mathbb{R}_h and \mathbb{R}_k denote the (discrete) collection of real numbers of the form x_n and t_j respectively and we shall refer to those sets and the cartesian product $\mathbb{R}_h \times \mathbb{R}_k$ as meshes. If ρ is any function defined on a mesh and taking values in either \mathbb{R} or \mathbb{R}^2 then we will refer to ρ as a mesh function. Obviously any function defined on a continuum \mathbb{R} or $\mathbb{R} \times \mathbb{R}$ (which we may call continuum functions) can also be considered as a mesh function by restricting its domain to the mesh. If ρ is a mesh function on $\mathbb{R}_h \times \mathbb{R}_k$ then we will ease some notation by writing $\rho_{n,j} := \rho(x_n, t_j)$ and $\rho_j := \rho(\cdot, t_j)$.

We shall make use of two discrete derivative operators D_+ , D_- that "differentiate" mesh functions ρ defined on \mathbb{R}_h (and hence they can also differentiate continuum functions ρ defined on \mathbb{R}). The operators are given by

$$D_+ \rho(x) = \frac{\rho(x+h) - \rho(x)}{h} \quad D_- \rho(x) = \frac{\rho(x) - \rho(x-h)}{h}$$

We will also sometimes use shifting operators E and E^{-1} given by

$$(E\rho)(x) = \rho(x + h) \quad (E^{-1}\rho)(x) = \rho(x - h)$$

These two operators will only act on the x variable of our functions $\rho(x, t)$. We will also use the operator $D_{t,+}\eta(t) = \frac{\eta(y+k)-\eta(t)}{k}$. The following properties of D_+ , D_- , $D_{t,+}$, E , and E^{-1} are immediate consequences of their definition:

1. If $\rho = \rho(x, t)$ then the operators D_+ , D_- , $D_{t,+}$ and E all commute when acting on $\rho(x, t)$.
2. $D_+(\nu \cdot \rho)(x_n) = \nu(x_n)D_+\rho(x_n) + (E\rho)(x_n)D_+\nu(x_n)$
3. $D_-(\nu \cdot \rho)(x_n) = \nu(x_n)D_-\rho(x_n) + (E^{-1}\rho)(x_n)D_-\nu(x_n)$
4. For any $n \in \mathbb{N}$ we have

$$D_+^n(\rho \cdot \nu \cdot \xi) = \sum_{i_1+i_2+i_3=n} c_{i_1,i_2,i_3} (E^{i_2+i_3} D_+^{i_1} \rho) \cdot (E^{i_3} D_+^{i_2} \nu) \cdot (D_+^{i_3} \xi)$$

for some constants $c_{i_1,i_2,i_3} \in \mathbb{N}$.

5. If a continuum function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable on \mathbb{R} then for each $x_n \in \mathbb{R}_h$ there exists $x \in \mathbb{R}$ where $x_n \leq x \leq x_{n+1}$ such that we have $D_+\rho(x_n) = \frac{d}{dx}\rho(x)$.

In order to solve the system (2.1), (2.2) we shall consider the following system of difference schemes, which are discrete versions of (2.1) and (2.2)

$$\begin{aligned} D_{t,+}(u_2)_j - D_+D_-(u_1)_{j+1} - \mu [(u_1)_j^2(u_1)_{j+1} + (u_1)_j(u_2)_j(u_2)_{j+1} + (f_2)_j^2(u_1)_j + 3(f_1)_j(u_1)_j(u_1)_{j+1} \\ + (f_1)_j(u_2)_j(u_2)_{j+1} + 2(f_2)_j(u_1)_j(u_2)_{j+1} + 3(f_1)_j^2(u_1)_j + 2(f_1)_j(f_2)_j(u_2)_j] - (g_1)_j = 0 \end{aligned} \quad (2.3)$$

$$\begin{aligned} D_{t,+}(u_1)_j + D_+D_-(u_2)_{j+1} + \mu [(u_1)_j(u_2)_j(u_1)_{j+1} + (u_2)_j^2(u_2)_{j+1} + 2(u_2)_j(f_1)_j(u_1)_{j+1} \\ + (u_1)_j(f_2)_j(u_1)_{j+1} + 3(u_2)_j(f_2)_j(u_2)_{j+1} + 2(f_1)_j(f_2)_j(u_1)_j + (f_1)_j^2(u_2)_j + 3(f_2)_j^2(u_2)_j] + (g_2)_j = 0 \end{aligned} \quad (2.4)$$

In order to rewrite this system of difference schemes in a more convenient and concise form we introduce a linear operator Q_j on pairs of mesh functions $(\rho_1(x_n), \rho_2(x_n))$ given by

$$\begin{aligned}
Q_j(\rho_1, \rho_2) := & (D_+ D_- \rho_2 + \mu(u_1)_j (u_2)_j \rho_1 + \mu(u_2)_j^2 \rho_2 + 2\mu(u_2)_j (f_1)_j \rho_1 \\
& + \mu(u_1)_j (f_2)_j \rho_1 + 3\mu(u_2)_j (f_2)_j \rho_2, -D_+ D_- \rho_1 - \mu(u_1)_j^2 \rho_1 - \mu(u_1)_j (u_2)_j \rho_2 \\
& - 3\mu(f_1)_j (u_1)_j \rho_1 - \mu(f_1)_j (u_2)_j \rho_2 - 2\mu(f_2)_j (u_1)_j \rho_2)
\end{aligned} \tag{2.5}$$

Then the system (2.3), (2.4) can be written in a shorter form as

$$\begin{aligned}
(I + kQ_j)((u_1)_{j+1}, (u_2)_{j+1}) = & ((u_1)_j, (u_2)_j) - k\mu((f_1)_j (f_2)_j (u_1)_j + (f_1)_j^2 (u_2)_j + 3(f_2)_j^2 (u_2)_j, \\
& -(f_2)_j^2 (u_1)_j - 3(f_1)_j^2 (u_1)_j - 2(f_1)_j (f_2)_j (u_2)_j) - k((g_2)_j, -(g_1)_j)
\end{aligned} \tag{2.6}$$

where $g_j \in S^{-\infty}(\mathbb{R})$ is considered as a mesh function. The task behind solving (2.6) then is to show that one can invert the operator $I + kQ_j$, at least for some finite amount of time. The invertibility will be possible only in certain function spaces, therefore we will now introduce an appropriate space.

Suppose $u(x_n) = (u_1(x_n), u_2(x_n))$ and $v(x_n) = (v_1(x_n), v_2(x_n))$ are mesh functions where $u_l, v_l : \mathbb{R}_h \rightarrow \mathbb{R}$ for $l = 1, 2$. We consider the discrete inner products given by

$$\begin{aligned}
(u_1, v_1)_{L_h^2} &= \sum_{-\infty}^{\infty} u_1(x_n) v_1(x_n) h \\
(u_1, v_1)_{S_h} &= (\langle x \rangle u_1, \langle x \rangle v_1)_{L_h^2} + (\langle x \rangle D_+ u_1, \langle x \rangle D_+ v_1)_{L_h^2} + (D_+^2 u_1, D_+^2 v_1)_{L_h^2} \\
(u, v)_{L_h^2} &= (u_1, v_1)_{L_h^2} + (u_2, v_2)_{L_h^2} \\
(u, v)_{S_h} &= (u_1, v_1)_{S_h} + (u_2, v_2)_{S_h}
\end{aligned}$$

where $\langle x \rangle = \sqrt{x^2 + 1}$, and we define the corresponding norms and Hilbert spaces,

$$\|u\|_{L_h^2}^2 = (u, u)_{L_h^2} \quad L_h^2 = \left\{ u(x_n) \text{ mesh functions on } \mathbb{R}_h : \|u\|_{L_h^2} < \infty \right\}$$

$$\|u\|_{S_h}^2 = (u, u)_{S_h} \quad S_h = \left\{ u(x_n) \text{ mesh functions on } \mathbb{R}_h : \|u\|_{S_h} < \infty \right\}$$

where we shall understand from the context whether L_h^2 and S_h refers to single-valued or double-valued mesh functions. From the definitions of the L_h^2 inner product and its norm we have the properties:

$$1. \|E\rho\|_{L_h^2} = \|\rho\|_{L_h^2}$$

2. If $\rho, \nu, D_- \rho, D_+ \nu \in L_h^2$ then $(D_+ \nu, \rho)_{L_h^2} = -(\nu, D_- \rho)_{L_h^2}$
3. For any $j, k \in \mathbb{N}$ we have $\|D_+^j D_-^k \rho\|_{L_h^2} \leq C \|\rho\|_{L_h^2}$ where C is a constant depending on h .
4. For any $l, N \in \mathbb{N}$ and $0 \leq j \leq N$ we have $\left\| \left(D_+^j \langle x \rangle^N \right) E^l \rho \right\|_{L_h^2} \leq C \left\| \langle x \rangle^{N-j} \rho \right\|_{L_h^2}$ where C is independent of h .

The first and second properties follow from simply reindexing and/or rearranging terms in the summation. The third property just requires use of the triangle inequality on each summand of $D_+^j D_-^k u$. For the fourth property we write the definition of D_+ and $\langle x \rangle$ and use the fact that $h \in [0, 1]$.

2.2 Preliminary Lemmas

The following proposition will be used frequently and often without reference. For the proof we refer the reader to [9] appendix B.

Proposition 2.1 *Let $N, n \in \mathbb{N}$, $T > 0$, $0 < h_1, k_1 < 1$ and let $g \in S^{-\infty}(\mathbb{R} \times [-c, \infty))$ for some $c > 0$. There exists $C_{N,n} > 0$ such that $\left\| \langle x \rangle^N D_+^n g_j \right\|_{L_h^2} < C_{N,n}$ for each $0 < h \leq h_1$, $0 < k \leq k_1$, and $0 \leq t_j \leq T$.*

Another simple but important fact that we will frequently use is the following:

If $f \in C^\infty(\mathbb{R} \times (-\infty, \infty) \rightarrow \mathbb{R})$ satisfies the property that for every $n \in \mathbb{N}$ and for every compact interval $J \subset (-\infty, \infty)$ we have $\frac{d^n}{dx^n} f(x, t) = O(|x|^{-n})$ uniformly for $t \in J$ then for each $t_j \in J$ and for all $x \in \mathbb{R}$ we have

$$\left| \frac{d^n}{dx^n} f(x, t_j) \right| \leq C |\langle x \rangle^{-n}|$$

where $C > 0$ is independent of k and j (but C might depend on J). This statement follows directly from the definitions of O and $\langle x \rangle$.

The discrete Sobolev inequalities stated below will allow us to prove that the operators $I + kQ_j$ for $j \in \mathbb{N}$ are bounded below and are thus invertible. These inequalities are stated and proven in [9] appendix B.

As a notational remark, from now on we will let C denote a constant whose value might change between consecutive inequalities but the variables that it depends on will often be noted by its indices for example as $C_{n,j,h}$ means some constant depending on n, j , and h .

Lemma 2.2 *For every $n \in \mathbb{N}$ there exists $C_n > 0$ such that for every $h > 0$ and for every mesh function $u : \mathbb{R}_h \rightarrow \mathbb{R}$ we have*

$$1. \quad \|D_+^k u\|_{L_h^2} \leq C_n \left(\|u\|_{L_h^2} + \|D_+^n u\|_{L_h^2} \right) \quad \text{for } 0 \leq k \leq n$$

$$2. \quad \|D_+^k u\|_\infty \leq C_n \left(\|u\|_{L_h^2} + \|D_+^n u\|_{L_h^2} \right) \quad \text{for } 0 \leq k < n$$

Corollary 2.3 For all $N, l, j \in \mathbb{N}$, there exists $C_{N,j,l} > 0$ such that for any $h \in (0, 1)$ and for all mesh functions $u : \mathbb{R}_h \rightarrow \mathbb{R}$ we have

$$1. \quad \|\langle x \rangle^N D_+^j u\|_{L_h^2} \leq C_{N,j,l} \left(\|\langle x \rangle^N u\|_{L_h^2} + \|\langle x \rangle^N D_+^{j+l} u\|_{L_h^2} \right) \quad \text{for } l \geq 0$$

$$2. \quad \|\langle x \rangle^N D_+^j u\|_\infty \leq C_{N,j,l} \left(\|\langle x \rangle^N u\|_{L_h^2} + \|\langle x \rangle^N D_+^{j+l} u\|_{L_h^2} \right) \quad \text{for } l \geq 1$$

The next lemma will allow us to prove that the solutions stay bounded for finite time with respect to the Schwartz semi-norms. The proof can be found in [15].

Lemma 2.4 Suppose P, Q are $C^1(\mathbb{R})$, nondecreasing, positive functions, $\Delta t > 0$, and for each $j \in \mathbb{N}$ we have $t_j := j\Delta t$. Let $\eta : [0, T_0] \rightarrow \mathbb{R}$ be an arbitrary function satisfying

$$\frac{\eta_{j+1} - \eta_j}{\Delta t} \leq P(\eta_j)\eta_{j+1} + Q(\eta_j) \quad (2.7)$$

for each $t_j, t_{j+1} \in [0, T_0]$ where $\eta_j := \eta(t_j)$, and suppose that $\eta_0 \leq K$ for some $K > 0$. Then there exists $0 < T \leq T_0$ and $L, \epsilon > 0$ all three depending on K, P , and Q such that if $\Delta t < \epsilon$ then $\eta_j \leq L$ for each j where $t_j \leq T$. Moreover, if P and Q are constants then we may take $T = T_0$.

2.3 Finite Time Existence for Discrete Generalized NLS Equation in S_h

We will now prove finite time existence for (2.6). The following lemma is the key estimate for establishing invertibility of the operator $I + kQ_j$ in the space S_h .

Lemma 2.5 Suppose $T > 0$, $h, k \in (0, 1)$ and that for each j where $t_j \in [0, T]$ we have a given mesh function $u_j : \mathbb{R}_h \rightarrow \mathbb{R}^2$. Define the operators Q_j as in (2.5). Then there exists $C > 0$ depending only on f and T but not on h, k, j , or the mesh functions u_j such that for any mesh function $u : \mathbb{R}_h \rightarrow \mathbb{R}^2$ the inequality

$$(Q_j u, u)_{S_h} \geq -C \|u\|_{S_h}^2 \left(1 + \|u_j\|_{S_h}^2 \right) \quad (2.8)$$

holds for each j where $0 \leq t_j \leq T$.

Proof of Lemma 2.5 We expand the left side of (2.8) by using the definition of Q_j and $(\cdot, \cdot)_{S_h}$ and obtain a sum of terms which can be bounded by the right side of (2.8). We will now show how to bound those terms by the right side of (2.8) for some appropriate constant C . Upon adding all the inequalities we will obtain inequality (2.8). Throughout we will denote $u(x_n) = (u_1(x_n), u_2(x_n))$ and $u_j = ((u_1)_j, (u_2)_j)$.

Estimate for Term $(D_+ D_- u_2, u_1)_{S_h} - (D_+ D_- u_1, u_2)_{S_h}$:

$$\begin{aligned}
(D_+ D_- u_2, u_1)_{S_h} - (D_+ D_- u_1, u_2)_{S_h} &= \left(D_+ D_- u_2, \langle x \rangle^2 u_1 \right)_{L_h^2} - \left(D_+ D_- u_1, \langle x \rangle^2 u_2 \right)_{L_h^2} \\
&\quad + \left(D_+^2 D_- u_2, \langle x \rangle^2 D_+ u_1 \right)_{L_h^2} - \left(D_+^2 D_- u_1, \langle x \rangle^2 D_+ u_2 \right)_{L_h^2} \\
&\quad + \left(D_+^3 D_- u_2, D_+^2 u_1 \right)_{L_h^2} - \left(D_+^3 D_- u_1, D_+^2 u_2 \right)_{L_h^2} \\
&= - \left(D_+ u_2, D_+ \langle x \rangle^2 E u_1 \right)_{L_h^2} - \left(D_+ u_2, \langle x \rangle^2 D_+ u_1 \right)_{L_h^2} \\
&\quad + \left(D_+ u_1, D_+ \langle x \rangle^2 E u_2 \right)_{L_h^2} + \left(D_+ u_1, \langle x \rangle^2 D_+ u_2 \right)_{L_h^2} \\
&\quad - \left(D_+^2 u_2, D_+ \langle x \rangle^2 D_+ u_1 \right)_{L_h^2} - \left(D_+^2 u_2, E \langle x \rangle^2 D_+^2 u_1 \right)_{L_h^2} \\
&\quad + \left(D_+^2 u_1, D_+ \langle x \rangle^2 D_+ u_2 \right)_{L_h^2} + \left(D_+^2 u_1, E \langle x \rangle^2 D_+^2 u_2 \right)_{L_h^2} \\
&\quad - \left(D_+^3 u_2, D_+^3 u_1 \right)_{L_h^2} + \left(D_+^3 u_1, D_+^3 u_2 \right)_{L_h^2} \\
&= - \left(D_+ u_2, D_+ \langle x \rangle^2 E u_1 \right)_{L_h^2} + \left(D_+ u_1, D_+ \langle x \rangle^2 E u_2 \right)_{L_h^2} \\
&\quad - \left(D_+^2 u_2, D_+ \langle x \rangle^2 D_+ u_1 \right)_{L_h^2} + \left(D_+^2 u_1, D_+ \langle x \rangle^2 D_+ u_2 \right)_{L_h^2} \\
&\geq -C \left[\|D_+ u_2\|_{L_h^2} \|\langle x \rangle u_1\|_{L_h^2} + \|D_+ u_1\|_{L_h^2} \|\langle x \rangle u_2\|_{L_h^2} \right. \\
&\quad \left. + \|D_+^2 u_2\|_{L_h^2} \|\langle x \rangle D_+ u_1\|_{L_h^2} + \|D_+^2 u_1\|_{L_h^2} \|\langle x \rangle D_+ u_2\|_{L_h^2} \right] \\
&\geq -C \|u_1\|_{S_h} \|u_2\|_{S_h} \geq -C \|u\|_{S_h}^2
\end{aligned}$$

Estimate for Term $(3\mu(u_2)_j(f_2)_j u_2, u_1)_{S_h}$:

By definition of the norm we have

$$\begin{aligned}
(3\mu(u_2)_j(f_2)_j u_2, u_1)_{S_h} &= (3\mu \langle x \rangle (u_2)_j(f_2)_j u_2, \langle x \rangle u_1)_{L_h^2} + (3\mu \langle x \rangle D_+ ((u_2)_j(f_2)_j u_2), \langle x \rangle D_+ u_1)_{L_h^2} \\
&\quad + (3\mu D_+^2 ((u_2)_j(f_2)_j u_2), D_+^2 u_1)_{L_h^2}
\end{aligned} \tag{2.9}$$

For the first term of (2.9) we have the bound

$$\begin{aligned}
(3\mu \langle x \rangle (u_2)_j (f_2)_j u_2, \langle x \rangle u_1)_{L_h^2} &\geq -C \| (f_2)_j \|_\infty \| (u_2)_j \|_\infty \| \langle x \rangle u_2 \|_{L_h^2} \| \langle x \rangle u_1 \|_{L_h^2} \\
&\geq -C \| (u_2)_j \|_{S_h} \| u_2 \|_{S_h} \| u_1 \|_{S_h} \\
&\geq -C \| u_j \|_{S_h} \| u \|_{S_h}^2 \geq -C \| u \|_{S_h}^2 \left(1 + \| u_j \|_{S_h}^2 \right)
\end{aligned}$$

For the second term of (2.9) we use the product rule for D_+ to obtain

$$\begin{aligned}
&(3\mu \langle x \rangle D_+ ((u_2)_j (f_2)_j u_2), \langle x \rangle D_+ u_1)_{L_h^2} \\
&= \sum_{i_1+i_2+i_3=1} c_{i_1,i_2,i_3} (E^{i_2+i_3} D_+^{i_1} (u_2)_j E^{i_3} D_+^{i_2} (f_2)_j D_+^{i_3} u_2, \langle x \rangle D_+ u_1)_{L_h^2}
\end{aligned}$$

For $i_1 = 1$ and other indices zero we have

$$\begin{aligned}
3\mu c_1 (\langle x \rangle (f_2)_j u_2 D_+ (u_2)_j, \langle x \rangle D_+ u_1)_{L_h^2} &\geq -C \| (f_2)_j \|_\infty \| D_+ (u_2)_j \|_\infty \| \langle x \rangle u_2 \|_{L_h^2} \| \langle x \rangle D_+ u_1 \|_{L_h^2} \\
&\geq -C \| (u_2)_j \|_{S_h} \| u_2 \|_{S_h} \| u_1 \|_{S_h} \geq -C \| u_j \|_{S_h} \| u \|_{S_h}^2 \\
&\geq -C \| u \|_{S_h}^2 \left(1 + \| u_j \|_{S_h}^2 \right)
\end{aligned}$$

For $i_2 = 1$ and other indices zero we have

$$\begin{aligned}
3\mu c_2 (\langle x \rangle u_2 E(u_2)_j D_+ (f_2)_j, \langle x \rangle D_+ u_1)_{L_h^2} &\geq -C \| D_+ (f_2)_j \|_\infty \| E(u_2)_j \|_\infty \| \langle x \rangle u_2 \|_{L_h^2} \| \langle x \rangle D_+ u_1 \|_{L_h^2} \\
&\geq -C \left\| \frac{d}{dx} (f_2)_j \right\|_\infty \| (u_2)_j \|_\infty \| u_2 \|_{S_h} \| u_1 \|_{S_h} \\
&\geq -C \| (u_2)_j \|_{S_h} \| u \|_{S_h}^2 \geq -C \| u \|_{S_h}^2 \left(1 + \| u_j \|_{S_h}^2 \right)
\end{aligned}$$

For $i_3 = 1$ and other indices zero we have

$$\begin{aligned}
3\mu c_3 (\langle x \rangle E(u_2)_j E(f_2)_j D_+ u_2, \langle x \rangle D_+ u_1)_{L_h^2} &\geq -C \| E(u_2)_j \|_\infty \| E(f_2)_j \|_\infty \| \langle x \rangle D_+ u_2 \|_{L_h^2} \| \langle x \rangle D_+ u_1 \|_{L_h^2} \\
&\geq -C \| (u_2)_j \|_{S_h} \| u_2 \|_{S_h} \| u_1 \|_{S_h} \geq -C \| u \|_{S_h}^2 \| u_j \|_{S_h} \\
&\geq -C \| u \|_{S_h}^2 \left(1 + \| u_j \|_{S_h}^2 \right)
\end{aligned}$$

For the third term of (2.9) we again use the product rule and obtain the expression

$$(3\mu D_+^2((f_2)_j(u_2)_j u_2), D_+^2 u_1)_{L_h^2} = \sum_{i_1+i_2+i_3=2} c_{i_1,i_2,i_3} (E^{i_2+i_3} D_+^{i_1}(f_2)_j E^{i_3} D_+^{i_2}(u_2)_j D_+^{i_3} u_2, D_+^2 u_1)_{L_h^2}$$

For $i_1 = 2$ and other indices zero we have

$$\begin{aligned} ((u_2)_j u_2 D_+^2(f_2)_j, D_+^2 u_1)_{L_h^2} &\geq -C \|D_+^2(f_2)_j\|_\infty \|(u_2)_j\|_\infty \|u_2\|_{L_h^2} \|D_+^2 u_1\|_{L_h^2} \\ &\geq -C \left\| \frac{d^2}{dx^2}(f_2)_j \right\|_\infty \|(u_2)_j\|_{S_h} \|u_2\|_{S_h} \|u_1\|_{S_h} \geq -C \|u_j\|_{S_h} \|u\|_{S_h}^2 \\ &\geq -C \|u\|_{S_h}^2 \left(1 + \|u_j\|_{S_h}^2 \right) \end{aligned}$$

For $i_2 = 2$ and other indices zero we have

$$\begin{aligned} (u_2 E^2(f_2)_j D_+^2(u_2)_j, D_+^2 u_1)_{L_h^2} &\geq -C \|E^2(f_2)_j\|_\infty \|u_2\|_\infty \|D_+^2(u_2)_j\|_{L_h^2} \|D_+^2 u_1\|_{L_h^2} \\ &\geq -C \|u_2\|_{S_h} \|(u_2)_j\|_{S_h} \|u_1\|_{S_h} \geq -C \|u\|_{S_h}^2 \left(1 + \|u_j\|_{S_h}^2 \right) \end{aligned}$$

For $i_3 = 2$ and other indices zero we have

$$\begin{aligned} (E^2(f_2)_j E^2(u_2)_j D_+^2 u_2, D_+^2 u_1)_{L_h^2} &\geq -C \|E^2(f_2)_j\|_\infty \|E^2(u_2)_j\|_\infty \|D_+^2 u_2\|_{L_h^2} \|D_+^2 u_1\|_{L_h^2} \\ &\geq -C \|(u_2)_j\|_{S_h} \|u_2\|_{S_h} \|u_1\|_{S_h} \geq -C \|u\|_{S_h}^2 \left(1 + \|u_j\|_{S_h}^2 \right) \end{aligned}$$

For $i_1 = i_2 = 1$ and $i_3 = 0$ we have

$$\begin{aligned} (u_2 E D_+(f_2)_j D_+(u_2)_j, D_+^2 u_1)_{L_h^2} &\geq -C \|ED_+(f_2)_j\|_\infty \|u_2\|_\infty \|D_+(u_2)_j\|_{L_h^2} \|D_+^2 u_1\|_{L_h^2} \\ &\geq -C \|u_2\|_{S_h} \|(u_2)_j\|_{S_h} \|u_1\|_{S_h} \geq -C \|u\|_{S_h}^2 \left(1 + \|u_j\|_{S_h}^2 \right) \end{aligned}$$

For $i_1 = i_3 = 1$ and $i_2 = 0$ we have

$$\begin{aligned} (ED_+(f_2)_j E(u_2)_j D_+ u_2, D_+^2 u_1)_{L_h^2} &\geq -C \|ED_+(f_2)_j\|_\infty \|E(u_2)_j\|_\infty \|D_+ u_2\|_{L_h^2} \|D_+^2 u_1\|_{L_h^2} \\ &\geq -C \|(u_2)_j\|_{S_h} \|u_2\|_{S_h} \|u_1\|_{S_h} \geq -C \|u\|_{S_h}^2 \left(1 + \|u_j\|_{S_h}^2 \right) \end{aligned}$$

For $i_2 = i_3 = 1$ and $i_1 = 0$ we have

$$\begin{aligned} (E^2(f_2)_j E D_+(u_2)_j D_+ u_2, D_+^2 u_1)_{L_h^2} &\geq -C \|E^2(f_2)_j\|_\infty \|ED_+(u_2)_j\|_\infty \|D_+ u_2\|_{L_h^2} \|D_+^2 u_1\|_{L_h^2} \\ &\geq -C \|(u_2)_j\|_{S_h} \|u_2\|_{S_h} \|u_1\|_{S_h} \geq -C \|u\|_{S_h}^2 \left(1 + \|u_j\|_{S_h}^2\right) \end{aligned}$$

One can similarly bound all other terms on the left side of (2.8) from the expression for Q_j and $(\cdot, \cdot)_{S_h}$ by using the same elementary inequalities as shown above. \square

Lemma 2.6 Suppose $K > 0$ and $u_0 = ((u_1)_0, (u_2)_0)$ where $(u_l)_0 \in S^{-\infty}(\mathbb{R} \rightarrow \mathbb{R})$ for $l = 1, 2$ satisfies the property that $\|u_0\|_{S_h} \leq K$ for each $h \in (0, 1)$. Then there exists $T, L, \epsilon > 0$ depending only on K such that if $k \in (0, \epsilon)$ and $h \in (0, 1)$ then the difference scheme (2.6) may be solved for each mesh function $u_j = ((u_1)_j, (u_2)_j)$ with $t_j \in [0, T]$. Moreover, we have that $\|u_j\|_{S_h} \leq L$ for each j where $t_j \in [0, T]$.

Proof of Lemma 2.6 Choose $T_0 > 0$ arbitrarily. Assume for now that the mesh functions $u_j = ((u_1)_j, (u_2)_j)$ are known for each $h, k \in (0, 1)$ and for $0 \leq t_j \leq T_0$. We will first construct the aforementioned T, L and an $\epsilon_0 > 0$ and show that the mesh functions u_j whose time mesh size satisfies $k \in (0, \epsilon_0)$ will satisfy the inequality $\|u_j\|_{S_h} \leq L$ for each $0 \leq t_j \leq T$.

Assume that $0 \leq t_j, t_{j+1} \leq T$. Taking inner product of (2.6) with u_{j+1} we obtain

$$\begin{aligned} ((I + kQ_j) u_{j+1}, u_{j+1})_{S_h} &= (u_j, u_{j+1})_{S_h} - k\mu \left(((f_1)_j (f_2)_j (u_1)_j + (f_1)_j^2 (u_2)_j + 3(f_2)_j^2 (u_2)_j, \right. \\ &\quad \left. -(f_2)_j^2 (u_1)_j - 3(f_1)_j^2 (u_1)_j - 2(f_1)_j (f_2)_j (u_2)_j, u_{j+1})_{S_h} \right) - k ((g_2)_j, -(g_1)_j, u_{j+1})_{S_h} \end{aligned}$$

and we may use Cauchy-Schwarz inequality on the right side and simply rewrite the left side to obtain

$$\|u_{j+1}\|_{S_h}^2 + k (Q_j u_{j+1}, u_{j+1})_{S_h} \leq \|u_{j+1}\|_{S_h} \cdot \|u_j\|_{S_h} + k C_{f,g,T_0} \cdot \|u_j\|_{S_h} \cdot \|u_{j+1}\|_{S_h} + k C_{g,T_0} \|u_{j+1}\|_{S_h} \quad (2.10)$$

By lemma 2.5 we may choose $C > 0$ such that

$$\|u_{j+1}\|_{S_h}^2 + k (Q_j u_{j+1}, u_{j+1})_{S_h} \geq \|u_{j+1}\|_{S_h}^2 \left[1 - kC(\|u_j\|_{S_h}^2 + 1) \right]$$

By combining (2.8) and (2.10) we thus obtain for $t_j, t_{j+1} \in [0, T_0]$ that

$$\|u_{j+1}\|_{S_h}^2 [1 - kC(\|u_j\|_{S_h}^2 + 1)] \leq \|u_{j+1}\|_{S_h} (\|u_j\|_{S_h} + kC \|u_j\|_{S_h} + kC)$$

or equivalently,

$$\frac{\|u_{j+1}\|_{S_h} - \|u_j\|_{S_h}}{k} \leq C (\|u_j\|_{S_h}^2 + 1) \|u_{j+1}\|_{S_h} + C (1 + \|u_j\|_{S_h})$$

which is an inequality of the form (2.7). Then by lemma 2.4 there exists $0 < T \leq T_0$ and $L, \epsilon_0 > 0$ depending on $K \geq \|u_0\|_{S_h}$ such that if $k \in (0, \epsilon_0)$ then $\|u_j\|_{S_h} \leq L$ for each j where $0 \leq t_j \leq T$. The T, L, ϵ_0 are independent of $h \in (0, 1)$ because the constant C is independent of h . Moreover, the T, L , and ϵ_0 depend only on $K, C, P(v) := v^2 + 1$, and $Q(v) := v + 1$ by lemma 2.4. Since K is given and C is determined by the given functions f and g and on the value of T_0 we may construct T, L , and ϵ_0 without assuming that u_j is constructed for $0 \leq t_j \leq T_0$.

Given T, L as constructed above it suffices to show that there exists $0 < \epsilon \leq \epsilon_0$ such that for any $h \in (0, 1)$ and $k \in (0, \epsilon)$ the difference scheme (2.6) may be solved for $u(x_n, t_j)$ where $(x_n, t_j) \in \mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ - the desired bound $\|u(\cdot, t_j)\|_{S_h} \leq L$ would follow automatically by our construction of T, L , and ϵ_0 given above.

Choose $\epsilon > 0$ so that $\epsilon C(L^2 + 1) < \frac{1}{2}$ and $0 < \epsilon \leq \epsilon_0$ and fix values for $h \in (0, 1)$ and $k \in (0, \epsilon)$. Suppose u_0, u_1, \dots, u_j are known for some $j \geq 0$. We will show that one may construct u_{j+1} as long as $t_{j+1} \leq T$. Define an operator $P_j := I + kQ_j$. Then by lemma 2.5 we have for any mesh function $u = u(x_n)$

$$(P_j u, u)_{S_h} \geq [1 - kC(\|u_j\|_{S_h}^2 + 1)] \|u\|_{S_h}^2 \geq [1 - \epsilon C(L^2 + 1)] \|u\|_{S_h}^2 \geq \frac{1}{2} \|u\|_{S_h}^2 \quad (2.11)$$

from which it easily follows that P_j is injective as an operator on mesh functions. It is also clear that for any mesh function $\rho = (\rho_1, \rho_2) \in S_h$ we have $\|P_j \rho\|_{S_h} \leq C_{f,g,h,k,u_j,T_0} \|\rho\|_{S_h}$ so that P_j is a bounded operator on S_h . In order to solve the difference scheme (2.6) it is enough to show that the operators P_j are surjective because then $P_j : S_h \rightarrow S_h$ would be a bijection so that by using (2.6) we could define

$$\begin{aligned} u_{j+1} := P_j^{-1} & [((u_1)_j, (u_2)_j) - k\mu ((f_1)_j(f_2)_j(u_1)_j + (f_1)_j^2(u_2)_j + 3(f_2)_j^2(u_2)_j , \\ & -(f_2)_j^2(u_1)_j - 3(f_1)_j^2(u_1)_j - 2(f_1)_j(f_2)_j(u_2)_j) - k ((g_2)_j, -(g_1)_j)] \end{aligned} \quad (2.12)$$

for $t_{j+1} \leq T$. First we shall prove that the image of P_j is closed.

Claim 1 The image of $P_j: S_h \rightarrow S_h$ is closed.

proof of claim 1 Suppose $P_j u_n \rightarrow v$ as $n \rightarrow \infty$. We will construct $u \in S_h$ such that $P_j u = v$.

Inequality (2.11) with $u_n - u_m$ implies, by using the Cauchy-Schwarz inequality on the left side, that for any $m, n \in \mathbb{N}$ we have

$$\|P_j(u_n - u_m)\|_{S_h} \geq \frac{1}{2} \|u_n - u_m\|_{S_h}$$

which implies that the sequence u_n is Cauchy, therefore by completeness of S_h the sequence u_n converges to some $u \in S_h$. Moreover,

$$\|P_j u - v\|_{S_h} \leq \|P_j u - P_j u_n\|_{S_h} + \|P_j u_n - v\|_{S_h} \leq \|P_j\| \|u - u_n\|_{S_h} + \|P_j u_n - v\|_{S_h}$$

and hence for n sufficiently large the right side can be made arbitrarily small so that $P_j u = v$. This concludes the proof of claim 1.

Claim 2 The operators $P_j: S_h \rightarrow S_h$ are surjective.

proof of claim 2 Since the image of P_j is closed we have that $S_h = \text{Im}(P_j) \oplus \text{Im}(P_j)^{\perp_{S_h}}$. Suppose there exists $v \in \text{Im}(P_j)^{\perp_{S_h}}$. Then $0 = (P_j v, v)_{S_h} \geq \frac{1}{2} \|v\|_{S_h}^2$ which implies that $v \equiv 0$. This proves surjectivity.

Therefore $P_j: S_h \rightarrow S_h$ is a bijection and hence we may solve the difference scheme (2.6) by defining u_{j+1} as in (2.12) as long as $t_j \leq T$ (recall that (2.11) fails for $t_j > T$ so that P_j would not be invertible after time T) however the desired bound $\|u_{j+1}\|_{S_h} \leq L$ would be true only if $t_{j+1} \leq T$.

□

Remark 2.7 A similar proof would also work even if the operators $I + kQ_j$ were unbounded as in the case of the KdV and modified KdV equation (see [3, 9, 15]).

3 Estimates for the Discrete Solutions

3.1 Schwartz Boundedness of Discrete Solutions

In this section we will show that the solutions to the discrete equation (2.6) constructed in lemma 2.6 are bounded in all discrete Schwartz norms $\|\langle x \rangle^N D_+^n \cdot\|_{L_h^2}$. This will follow by some induction arguments shown in next three lemmas.

Lemma 3.1 Let $u_0 = ((u_1)_0, (u_2)_0)$ where $(u_l)_0 \in S^{-\infty}(\mathbb{R})$ for $l = 1, 2$ and let $K, T, \epsilon > 0$ all be given as in lemma 2.6 and let $N, n \in \mathbb{N}$. Suppose that there exists a constant $C_{N,n} > 0$ independent of $h \in (0, 1)$, $k \in (0, \epsilon)$, and $t_j \in [0, T]$ such that if $u(x_n, t_j) = (u_1(x_n, t_j), u_2(x_n, t_j))$ is the solution of the difference scheme (2.6) defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ with initial condition u_0 then for $0 \leq t_j, t_{j+1} \leq T$ we have

$$\left(\langle x \rangle^{2N} D_+^n (Q_j u_{j+1}), D_+^n u_{j+1} \right)_{L_h^2} \geq -C_{N,n} \left(\left\| \langle x \rangle^N D_+^n u_{j+1} \right\|_{L_h^2}^2 + \left\| \langle x \rangle^N D_+^n u_j \right\|_{L_h^2}^2 + 1 \right) \quad (3.1)$$

Then there exists $0 < k_0 \leq \epsilon$ depending on N and n and there exists $C_{N,n} > 0$ such that if $h \in (0, 1)$, $k \in (0, k_0)$, and $u(x_n, t_j)$ is the solution of the difference scheme (2.6) defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ with initial condition u_0 then we have $\left\| \langle x \rangle^N D_+^n u_j \right\|_{L_h^2} < C_{N,n}$ for $t_j \in [0, T]$. Moreover, the constant $C_{N,n}$ is independent of the choice of h and k .

Proof of Lemma 3.1 First let us fix values for $h \in (0, 1)$ and $k \in (0, \epsilon)$ and let $u(x_n, t_j)$ be the solution of the difference scheme (2.6) defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ with initial condition u_0 . If $t_j, t_{j+1} \in [0, T]$ then we may apply D_+^n to both sides of (2.6) and take the inner product $\left(\langle x \rangle^{2N} \cdot, \cdot \right)_{L_h^2}$ of the resulting equation with $D_+^n u_{j+1}$ and use proposition 2.1, the product rule, and corollary 2.3 to obtain

$$\begin{aligned} & \left(\langle x \rangle^{2N} D_+^n (I + kQ_j) u_{j+1}, D_+^n u_{j+1} \right)_{L_h^2} = \left(\langle x \rangle^{2N} D_+^n u_j, D_+^n u_{j+1} \right)_{L_h^2} \\ & - k\mu \left(\langle x \rangle^{2N} D_+^n ((f_1)_j (f_2)_j (u_1)_j + (f_1)_j^2 (u_2)_j + 3(f_2)_j^2 (u_2)_j, \right. \\ & \left. - (f_2)_j^2 (u_1)_j - 3(f_1)_j^2 (u_1)_j - 2(f_1)_j (f_2)_j (u_2)_j, D_+^n u_{j+1} \right)_{L_h^2} - k \left(\langle x \rangle^{2N} D_+^n ((g_2)_j, -(g_1)_j), D_+^n u_{j+1} \right)_{L_h^2} \\ & \leq \left\| \langle x \rangle^N D_+^n u_{j+1} \right\|_{L_h^2} \left\| \langle x \rangle^N D_+^n u_j \right\|_{L_h^2} + kC_{f,g,K} \left\| \langle x \rangle^N D_+^n u_{j+1} \right\|_{L_h^2} \left(1 + \left\| \langle x \rangle^N D_+^n u_j \right\|_{L_h^2} \right) \\ & \leq \frac{1}{2} \left\| \langle x \rangle^N D_+^n u_{j+1} \right\|_{L_h^2}^2 + \frac{1}{2} \left\| \langle x \rangle^N D_+^n u_j \right\|_{L_h^2}^2 + kC_{N,n,f,g} \left(1 + \left\| \langle x \rangle^N D_+^n u_j \right\|_{L_h^2}^2 + \left\| \langle x \rangle^N D_+^n u_{j+1} \right\|_{L_h^2}^2 \right) \end{aligned} \quad (3.2)$$

We may then continue the left side of (3.2) by using (3.1) to obtain

$$\begin{aligned} & \left(\langle x \rangle^{2N} D_+^n (I + kQ_j) u_{j+1}, D_+^n u_{j+1} \right)_{L_h^2} = \left\| \langle x \rangle^N D_+^n u_{j+1} \right\|_{L_h^2}^2 + k \left(\langle x \rangle^{2N} D_+^n Q_j u_{j+1}, D_+^n u_{j+1} \right)_{L_h^2} \\ & \geq \left\| \langle x \rangle^N D_+^n u_{j+1} \right\|_{L_h^2}^2 - kC \left(\left\| \langle x \rangle^N D_+^n u_j \right\|_{L_h^2}^2 + \left\| \langle x \rangle^N D_+^n u_{j+1} \right\|_{L_h^2}^2 + 1 \right) \end{aligned} \quad (3.3)$$

By combining (3.2) and (3.3) we obtain

$$\frac{1}{k} \left(\left\| \langle x \rangle^N D_+^n u_{j+1} \right\|_{L_h^2}^2 - \left\| \langle x \rangle^N D_+^n u_j \right\|_{L_h^2}^2 \right) \leq C \left(\left\| \langle x \rangle^N D_+^n u_{j+1} \right\|_{L_h^2}^2 + \left\| \langle x \rangle^N D_+^n u_j \right\|_{L_h^2}^2 + 1 \right)$$

which is an inequality of the form (2.7) where the right side is linear. Now we may invoke lemma 2.4 to obtain the existence of $k_0 > 0$ and $C_{N,n} > 0$ such that if $k \in (0, k_0)$ and $t_j \in [0, T]$ then

$$\left\| \langle x \rangle^N D_+^n u_j \right\|_{L_h^2} < C_{N,n}$$

and since C is independent of h and k we also have that C is independent of h and k . Thus we have proven that for each $0 < h \leq 1$, $0 < k \leq k_0$, $0 \leq t_j \leq T$ and solution $u(x_n, t_j)$ of (2.6) defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ with initial data u_0 we have the inequality $\left\| \langle x \rangle^N D_+^n u_j \right\|_{L_h^2} < C_{N,n}$. Moreover since C depends on N and n it follows that k_0 also depends on N and n . \square

Lemma 3.2 *Let $u_0 = ((u_1)_0, (u_2)_0)$ where $(u_l)_0 \in S^{-\infty}(\mathbb{R})$ for $l = 1, 2$ and let $K, T, L, \epsilon > 0$ all be given as in lemma 2.6 and let $n \in \mathbb{N}$. Then there exists $0 < k_0 \leq \epsilon$ depending on n and there exists $C_n > 0$ such that if $h \in (0, 1)$, $k \in (0, k_0)$, and $u(x_n, t_j) = (u_1(x_n, t_j), u_2(x_n, t_j))$ is the solution of the difference scheme (2.6) defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ with initial condition u_0 then we have $\left\| \langle x \rangle^N D_+^n u_j \right\|_{L_h^2} < C_n$ for $t_j \in [0, T]$. Moreover, the constant C_n is independent of the choice of h and k .*

Proof of Lemma 3.2 By lemma 2.6 the statement is true for $n = 0, 1$ by taking $k_0 = \epsilon$ and $C_n = L$. We shall prove the statement by induction on n . Assume it is true for all $l \leq n - 1$. We will prove that it is true for $l = n$. First we shall use the inductive hypothesis to prove some slightly weaker claims which we give below.

Claim 1 There exists $0 < \epsilon_0 \leq \epsilon$ and $C_n > 0$ such that if $h \in (0, 1)$ and $k \in (0, \epsilon_0)$ and $u(x_n, t_j)$ is the solution of the difference scheme (2.6) defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ with initial condition u_0 then we have $\left\| D_+^n u_j \right\|_{L_h^2} \leq C_n$ for $t_j \in [0, T]$.

proof of claim 1 By lemma 3.1 it suffices to prove that there exists a constant $C_n > 0$ independent of $h \in (0, 1)$, $k \in (0, \epsilon)$, and $t_j \in [0, T]$ such that if $u(x_n, t_j)$ is the solution of the difference scheme (2.6) defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ with initial condition u_0 then for $0 \leq t_j, t_{j+1} \leq T$ we have

$$(D_+^n(Q_j u_{j+1}), D_+^n u_{j+1})_{L_h^2} \geq -C_n \left(\left\| D_+^n u_{j+1} \right\|_{L_h^2}^2 + \left\| D_+^n u_j \right\|_{L_h^2}^2 + 1 \right)$$

To this end we shall fix values for $h \in (0, 1)$ and $k \in (0, \epsilon)$ and a solution $u(x_n, t_j)$ of the difference scheme (2.6) defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ with initial condition u_0 . In order to prove that the above estimate holds for some constant $C_n > 0$ we simply prove that the estimate can be made for each term of Q_j and then by adding all these estimates we will obtain the estimate for Q_j . These estimates are analogous to those given in lemma 2.5 however we will also use the inductive hypothesis.

For simplicity we will ignore all occurrences of the shift E . Here are the necessary estimates.

Estimate for Term $(D_+^n(D_+D_-(u_2)_{j+1}), D_+^n(u_1)_{j+1})_{L_h^2} - (D_+^n(D_+D_-(u_1)_{j+1}), D_+^n(u_2)_{j+1})_{L_h^2}$:

$$\begin{aligned} & (D_+^n(D_+D_-(u_2)_{j+1}), D_+^n(u_1)_{j+1})_{L_h^2} - (D_+^n(D_+D_-(u_1)_{j+1}), D_+^n(u_2)_{j+1})_{L_h^2} \\ &= - (D_+^{n+1}(u_2)_{j+1}, D_+^{n+1}(u_1)_{j+1})_{L_h^2} + (D_+^{n+1}(u_1)_{j+1}, D_+^{n+1}(u_2)_{j+1})_{L_h^2} = 0 \end{aligned}$$

Estimate for Term $(D_+^n((u_1)_j(u_2)_j(u_1)_{j+1}), D_+^n(u_1)_{j+1})_{L_h^2}$:

By the product rule we obtain,

$$\begin{aligned} & (D_+^n((u_1)_j(u_2)_j(u_1)_{j+1}), D_+^n(u_1)_{j+1})_{L_h^2} \\ &= \sum_{i_1+i_2+i_3=n} c_{i_1,i_2,i_3} (D_+^{i_1}(u_1)_j D_+^{i_2}(u_2)_j D_+^{i_3}(u_1)_{j+1}, D_+^n(u_1)_{j+1})_{L_h^2} \end{aligned}$$

If $2 \leq i_1 + i_2 \leq n$ and $i_1 \leq i_2$ then we have the estimate,

$$\begin{aligned} & (D_+^{i_1}(u_1)_j D_+^{i_2}(u_2)_j D_+^{i_3}(u_1)_{j+1}, D_+^n(u_1)_{j+1})_{L_h^2} \\ & \geq -C \|D_+^{i_1}(u_1)_j\|_\infty \|D_+^{i_2}(u_2)_j\|_{L_h^2} \|D_+^{i_3}(u_1)_{j+1}\|_\infty \|D_+^n(u_1)_{j+1}\|_{L_h^2} \\ & \geq -C \left(\|u_2\|_{L_h^2} + \|D_+^n(u_2)\|_{L_h^2} \right) \|D_+^n(u_1)_{j+1}\|_{L_h^2} \\ & \geq -C \left(1 + \|D_+^n(u_1)_{j+1}\|_{L_h^2}^2 + \|D_+^n(u_2)_j\|_{L_h^2}^2 \right) \\ & \geq -C \left(1 + \|D_+^n u_{j+1}\|_{L_h^2}^2 + \|D_+^n u_j\|_{L_h^2}^2 \right) \end{aligned}$$

Also, if instead we had $2 \leq i_1 + i_2 \leq n$ and $i_2 \leq i_1$ then the same estimates work by just switching $(u_1)_j$ and $(u_2)_j$.

For the term where $i_1 + i_2 = 1$ and $i_1 = 1$ we have the estimate,

$$\begin{aligned}
& \left((u_2)_j D_+(u_1)_j D_+^{n-1}(u_1)_{j+1}, D_+^n(u_1)_{j+1} \right)_{L_h^2} \\
& \geq - \| (u_2)_j \|_\infty \| D_+^{n-1}(u_1)_{j+1} \|_\infty \| D_+(u_1)_j \|_{L_h^2} \| D_+^n(u_1)_{j+1} \|_{L_h^2} \\
& \geq -C \left(\| (u_1)_{j+1} \|_{L_h^2} + \| D_+^n(u_1)_{j+1} \|_{L_h^2} \right) \| D_+^n(u_1)_{j+1} \|_{L_h^2} \\
& \geq -C \left(1 + \| D_+^n(u_1)_{j+1} \|_{L_h^2}^2 \right) \geq -C \left(1 + \| D_+^n u_{j+1} \|_{L_h^2}^2 \right)
\end{aligned}$$

Also, if instead we had $i_1 + i_2 = 1$ and $i_2 = 1$ then the same estimates work by just switching $(u_1)_j$ and $(u_2)_j$.

For the term where $i_1 + i_2 = 0$ we have the estimate,

$$\begin{aligned}
\left((u_1)_j (u_2)_j D_+^n(u_1)_{j+1}, D_+^n(u_1)_{j+1} \right)_{L_h^2} & \geq - \| (u_1)_j \|_\infty \| (u_2)_j \|_\infty \| D_+^n(u_1)_{j+1} \|_{L_h^2}^2 \\
& \geq -C \left(\| u_j \|_{L_h^2} + \| D_+ u_j \|_{L_h^2} \right)^2 \| D_+^n u_{j+1} \|_{L_h^2}^2 \\
& \geq -C \| D_+^n u_j \|_{L_h^2}^2
\end{aligned}$$

All other terms can be similarly estimated. This concludes the proof of claim 1.

Claim 2 There exists $0 < \epsilon_1 \leq \epsilon_0$ and $C_n > 0$ such that if $h \in (0, 1)$ and $k \in (0, \epsilon_1)$ and $u(x_n, t_j)$ is the solution of the difference scheme (2.6) defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ with initial condition u_0 then we have $\| D_+^{n+1} u_j \|_{L_h^2} \leq C_n$ for $t_j \in [0, T]$.

proof of claim 2 As in the proof of claim 1 we see that by lemma 3.1 it suffices to prove that there exists a constant $C_n > 0$ independent of $h \in (0, 1)$, $k \in (0, \epsilon_0)$, and $t_j \in [0, T]$ such that if $u(x_n, t_j)$ is the solution of the difference scheme (2.6) defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ with initial condition u_0 then for $0 \leq t_j, t_{j+1} \leq T$ we have

$$(D_+^{n+1}(Q_j u_{j+1}), D_+^{n+1} u_{j+1})_{L_h^2} \geq -C_n \left(\| D_+^{n+1} u_{j+1} \|_{L_h^2}^2 + \| D_+^{n+1} u_j \|_{L_h^2}^2 + 1 \right)$$

To this end we shall again fix values for $h \in (0, 1)$ and $k \in (0, \epsilon_0)$ and a solution $u(x_n, t_j)$ to the solution of the difference scheme (2.6) defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ with initial condition u_0 . In order to prove that the above estimate holds for some constant $C_n > 0$ we simply prove that the estimate can be made for each term of Q_j and then by adding all these estimates we will obtain the estimate for Q_j . By using the inductive hypothesis and the result of claim 1, it is easily seen that the estimates shown in the proof of claim 1 can all be applied with n replaced by $n + 1$ and the

others can be similarly estimated. This concludes the proof of claim 2.

Now we return to the proof of the lemma in the case $l = n$. By using the same reasoning as in the above claims we see by lemma 3.1 that in order to construct $k_0 \in (0, \epsilon_2)$ and C_n it suffices to prove the estimate

$$\left(\langle x \rangle^2 D_+^n(Q_j u_{j+1}), D_+^n u_{j+1} \right)_{L_h^2} \geq -C_n \left(\|\langle x \rangle D_+^n u_{j+1}\|_{L_h^2}^2 + \|\langle x \rangle D_+^n u_j\|_{L_h^2}^2 + 1 \right)$$

This estimate will follow by adding all of the below estimates. In the below estimates we will use our inductive hypothesis and the estimates $\|D_+^{n+i} u_j\|_{L_h^2} \leq C_{n+i}$ for $i = 0, 1$. For conciseness we shall again only formulate estimates for a few terms because the rest can be bounded similarly.

Estimate for Term $\left(D_+ D_- D_+^n(u_2)_{j+1}, \langle x \rangle^2 D_+^n(u_1)_{j+1} \right)_{L_h^2} - \left(D_+ D_- D_+^n(u_1)_{j+1}, \langle x \rangle^2 D_+^n(u_2)_{j+1} \right)_{L_h^2} :$

$$\begin{aligned} & \left(D_+ D_- (D_+^n(u_2)_{j+1}), \langle x \rangle^2 D_+^n(u_1)_{j+1} \right)_{L_h^2} - \left(D_+ D_- (D_+^n(u_1)_{j+1}), \langle x \rangle^2 D_+^n(u_2)_{j+1} \right)_{L_h^2} \\ &= - \left(D_+^{n+1}(u_2)_{j+1}, E \langle x \rangle^2 D_+^{n+1}(u_1)_{j+1} \right)_{L_h^2} - \left(D_+^{n+1}(u_2)_{j+1}, D_+^n(u_1)_{j+1} D_+ \langle x \rangle^2 \right)_{L_h^2} \\ & \quad + \left(D_+^{n+1}(u_1)_{j+1}, E \langle x \rangle^2 D_+^{n+1}(u_2)_{j+1} \right)_{L_h^2} + \left(D_+^{n+1}(u_1)_{j+1}, D_+^n(u_2)_{j+1} D_+ \langle x \rangle^2 \right)_{L_h^2} \\ &= \left(D_+^{n+1}(u_1)_{j+1}, D_+^n(u_2)_{j+1} D_+ \langle x \rangle^2 \right)_{L_h^2} - \left(D_+^{n+1}(u_2)_{j+1}, D_+^n(u_1)_{j+1} D_+ \langle x \rangle^2 \right)_{L_h^2} \\ &\geq - \|D_+^{n+1}(u_1)_{j+1}\|_{L_h^2} \|\langle x \rangle D_+^n(u_2)_{j+1}\|_{L_h^2} - \|D_+^{n+1}(u_2)_{j+1}\|_{L_h^2} \|\langle x \rangle D_+^n(u_1)_{j+1}\|_{L_h^2} \\ &\geq -C \left(\|\langle x \rangle D_+^n(u_2)_{j+1}\|_{L_h^2} + \|\langle x \rangle D_+^n(u_1)_{j+1}\|_{L_h^2} \right) \geq -C \left(\|\langle x \rangle D_+^n u_{j+1}\|_{L_h^2}^2 + 1 \right) \end{aligned}$$

Estimate for Term $\left(\langle x \rangle D_+^n((u_1)_j(u_2)_j(u_1)_{j+1}), \langle x \rangle D_+^n(u_1)_{j+1} \right)_{L_h^2} :$

By the product rule we obtain,

$$\begin{aligned} & \left(\langle x \rangle D_+^n((u_1)_j(u_2)_j(u_1)_{j+1}), \langle x \rangle D_+^n(u_1)_{j+1} \right)_{L_h^2} \\ &= \sum_{i_1+i_2+i_3=n} c_{i_1, i_2, i_3} \left(\langle x \rangle D_+^{i_1}(u_1)_j D_+^{i_2}(u_2)_j D_+^{i_3}(u_1)_{j+1}, \langle x \rangle D_+^n(u_1)_{j+1} \right)_{L_h^2} \end{aligned}$$

If $2 \leq i_1 + i_2 \leq n$ and $i_1 \leq i_2$ then we have the estimate,

$$\begin{aligned} & \left(\langle x \rangle D_+^{i_1}(u_1)_j D_+^{i_2}(u_2)_j D_+^{i_3}(u_1)_{j+1}, \langle x \rangle D_+^n(u_1)_{j+1} \right)_{L_h^2} \\ &\geq -C \|D_+^{i_1}(u_1)_j\|_\infty \|D_+^{i_3}(u_1)_{j+1}\|_\infty \|\langle x \rangle D_+^{i_2}(u_2)_j\|_{L_h^2} \|\langle x \rangle D_+^n(u_1)_{j+1}\|_{L_h^2} \\ &\geq -C \|\langle x \rangle D_+^n(u_1)_{j+1}\|_{L_h^2} \geq -C \left(1 + \|\langle x \rangle D_+^n u_{j+1}\|_{L_h^2}^2 \right) \end{aligned}$$

Also, if instead we had $2 \leq i_1 + i_2 \leq n$ and $i_2 \leq i_1$ then the same estimates work by just switching $(u_1)_j$ and $(u_2)_j$.

For the term where $i_1 + i_2 = 1$ and $i_1 = 1$ we have the estimate,

$$\begin{aligned} & (\langle x \rangle (u_2)_j D_+(u_1)_j D_+^{n-1}(u_1)_{j+1}, \langle x \rangle D_+^n(u_1)_{j+1})_{L_h^2} \\ & \geq -\|\langle x \rangle (u_2)_j\|_\infty \|D_+(u_1)_j\|_\infty \|\langle x \rangle D_+^{n-1}(u_1)_{j+1}\|_{L_h^2} \|\langle x \rangle D_+^n(u_1)_{j+1}\|_{L_h^2} \\ & \geq -C \|\langle x \rangle D_+^n(u_1)_{j+1}\|_{L_h^2} \geq -C \left(1 + \|\langle x \rangle D_+^n u_{j+1}\|_{L_h^2}^2\right) \end{aligned}$$

Also, if instead we had $i_1 + i_2 = 1$ and $i_2 = 1$ then the same estimates work by just switching $(u_1)_j$ and $(u_2)_j$.

For the term where $i_1 + i_2 = 0$ we have the estimate,

$$(\langle x \rangle (u_1)_j (u_2)_j D_+^n(u_1)_{j+1}, \langle x \rangle D_+^n(u_1)_{j+1})_{L_h^2} \geq -\|\langle x \rangle (u_1)_j\|_\infty \|\langle x \rangle (u_2)_j\|_\infty \|D_+^n(u_1)_{j+1}\|_{L_h^2}^2 \geq -C$$

This concludes the proof of the lemma. \square

Lemma 3.3 *Let $u_0 = ((u_1)_0, (u_2)_0)$ where $(u_l)_0 \in S^{-\infty}(\mathbb{R})$ for $l = 1, 2$ and let $K, T, \epsilon > 0$ all be given as in lemma 2.6 and let $N, n \in \mathbb{N}$. Then there exists $0 < k_0 \leq \epsilon$ depending on N and n and there exists $C_{N,n} > 0$ such that if $h \in (0, 1)$, $k \in (0, k_0)$, and $u(x_n, t_j) = (u_1(x_n, t_j), u_2(x_n, t_j))$ is the solution of the difference scheme (2.6) defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ with initial condition u_0 then we have $\|\langle x \rangle^N D_+^n u_j\|_{L_h^2} < C_{N,n}$ for $t_j \in [0, T]$. Moreover, the constant $C_{N,n}$ is independent of the choice of h and k .*

Proof of Lemma 3.3 By lemma 3.2 the assertion is true for $N = 1$ and for all $n \in \mathbb{N}$. We prove the assertion by induction on N . Assume it is true for all $M \leq N - 1$ and for all $n \in \mathbb{N}$. We will show that for $M = N$ the statement is satisfied by all $n \in \mathbb{N}$ by induction on n .

Let us denote the value of k_0 corresponding to a particular value of N and n by $k_0(N, n)$. By construction of $k_0(N, n)$ for its known values we see that if $N_2 < N_1$ then $k_0(N_1, n) \leq k_0(N_2, n)$ and if $n_2 < n_1$ then $k_0(N, n_1) \leq k_0(N, n_2)$. Let $\epsilon_0 := k_0(N-1, 1)$. By use of lemma 3.1 and taking $\epsilon = \epsilon_0$ our statement for $n = 0$ will follow if we prove that there exists a constant $C_{N,0} > 0$ independent of $h \in (0, 1)$, $k \in (0, \epsilon_0)$, and $t_j \in [0, T]$ such that if $u(x_n, t_j)$ is the solution of the difference scheme (2.6) defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ with initial condition u_0 then for $0 \leq t_j, t_{j+1} \leq T$ we have

$$\left(\langle x \rangle^{2N} Q_j u_{j+1}, u_{j+1} \right)_{L_h^2} \geq -C_{N,0} \left(\left\| \langle x \rangle^N u_{j+1} \right\|_{L_h^2}^2 + \left\| \langle x \rangle^N u_j \right\|_{L_h^2}^2 + 1 \right)$$

To this end we shall fix values for $h \in (0, 1)$ and $k \in (0, \epsilon_0)$ and a solution $u(x_n, t_j)$ of the difference scheme (2.6) defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ with initial condition u_0 . In order to prove that the above estimate holds for some constant $C_{N,0} > 0$ we simply prove that the estimate can be made for each term of Q_j and then by adding all these estimates we will obtain the estimate for Q_j . We will again only show a few estimates since all can be done similarly and we will ignore all occurrences of the shift E for simplicity.

Estimate for Term $\left(D_+ D_- (u_2)_{j+1}, \langle x \rangle^{2N} (u_1)_{j+1} \right)_{L_h^2} - \left(D_+ D_- (u_1)_{j+1}, \langle x \rangle^{2N} (u_2)_{j+1} \right)_{L_h^2} :$

$$\begin{aligned} & \left(D_+ D_- (u_2)_{j+1}, \langle x \rangle^{2N} (u_1)_{j+1} \right)_{L_h^2} - \left(D_+ D_- (u_1)_{j+1}, \langle x \rangle^{2N} (u_2)_{j+1} \right)_{L_h^2} \\ &= \left(D_+ (u_1)_{j+1}, (u_2)_{j+1} D_+ \langle x \rangle^{2N} \right)_{L_h^2} - \left(D_+ (u_2)_{j+1}, (u_1)_{j+1} D_+ \langle x \rangle^{2N} \right)_{L_h^2} \\ &\geq -C \left\| \langle x \rangle^{N-1} D_+ (u_1)_{j+1} \right\|_{L_h^2} \left\| \langle x \rangle^N (u_2)_{j+1} \right\|_{L_h^2} - C \left\| \langle x \rangle^{N-1} D_+ (u_2)_{j+1} \right\|_{L_h^2} \left\| \langle x \rangle^N (u_1)_{j+1} \right\|_{L_h^2} \\ &\geq -C \left\| \langle x \rangle^N u_{j+1} \right\|_{L_h^2} \end{aligned}$$

Estimate for Term $\left(\langle x \rangle^N (u_1)_j (u_2)_j (u_1)_{j+1}, \langle x \rangle^N (u_1)_{j+1} \right)_{L_h^2} :$

$$\begin{aligned} \left(\langle x \rangle^N (u_1)_j (u_2)_j (u_1)_{j+1}, \langle x \rangle^N (u_1)_{j+1} \right)_{L_h^2} &\geq \left\| (u_1)_j \right\|_\infty \left\| (u_2)_j \right\|_\infty \left\| \langle x \rangle^N (u_1)_{j+1} \right\|_{L_h^2}^2 \\ &\geq -C \left\| \langle x \rangle^N u_{j+1} \right\|_{L_h^2}^2 \end{aligned}$$

This concludes the proof for the case $n = 0$.

Now we will assume that the statement is true for all $l \leq n - 1$ and we will prove that it is true for $l = n$. Let $\epsilon_1 := k_0(N - 1, n + 1)$. The proof will again follow from lemma 3.1 by taking $\epsilon = \epsilon_1$ if we can prove that there exists a constant $C_{N,n} > 0$ independent of $h \in (0, 1)$, $k \in (0, \epsilon_1)$, and $t_j \in [0, T]$ such that if $u(x_n, t_j)$ is the solution of the difference scheme (2.6) defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ with

initial condition u_0 then for $0 \leq t_j, t_{j+1} \leq T$ we have

$$\left(\langle x \rangle^{2N} D_+^n (Q_j u_{j+1}), D_+^n u_{j+1} \right)_{L_h^2} \geq -C_{N,n} \left(\left\| \langle x \rangle^N D_+^n u_{j+1} \right\|_{L_h^2}^2 + \left\| \langle x \rangle^N D_+^n u_j \right\|_{L_h^2}^2 + 1 \right)$$

To this end we shall fix values for $h \in (0, 1)$ and $k \in (0, \epsilon_1)$ and a solution $u(x_n, t_j)$ of the difference scheme (2.6) defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ with initial condition u_0 . In order to prove that the above estimate holds for some constant $C_{N,n} > 0$ we again prove that the estimate can be made for each term of Q_j and then by adding all these estimates we will obtain the estimate for Q_j . Here are estimates for some terms - the rest can be estimated similarly.

$$\underline{\text{Estimate for Term}} \left(D_+ D_- D_+^n (u_2)_{j+1}, \langle x \rangle^{2N} D_+^n (u_1)_{j+1} \right)_{L_h^2} - \left(D_+ D_- D_+^n (u_1)_{j+1}, \langle x \rangle^{2N} D_+^n (u_2)_{j+1} \right)_{L_h^2} :$$

$$\begin{aligned} & \left(D_+ D_- D_+^n (u_2)_{j+1}, \langle x \rangle^{2N} D_+^n (u_1)_{j+1} \right)_{L_h^2} - \left(D_+ D_- D_+^n (u_1)_{j+1}, \langle x \rangle^{2N} D_+^n (u_2)_{j+1} \right)_{L_h^2} \\ &= \left(D_+^{n+1} (u_1)_{j+1}, D_+^n (u_2)_{j+1} D_+ \langle x \rangle^{2N} \right)_{L_h^2} - \left(D_+^{n+1} (u_2)_{j+1}, D_+^n (u_1)_{j+1} D_+ \langle x \rangle^{2N} \right)_{L_h^2} \\ &\geq -C \left\| \langle x \rangle^{N-1} D_+^{n+1} (u_1)_{j+1} \right\|_{L_h^2} \left\| \langle x \rangle^N D_+^n (u_2)_{j+1} \right\|_{L_h^2} \\ &\quad - C \left\| \langle x \rangle^{N-1} D_+^{n+1} (u_2)_{j+1} \right\|_{L_h^2} \left\| \langle x \rangle^N D_+^n (u_1)_{j+1} \right\|_{L_h^2} \\ &\geq -C \left(\left\| \langle x \rangle^N D_+^n (u_1)_{j+1} \right\|_{L_h^2} + \left\| \langle x \rangle^N D_+^n (u_2)_{j+1} \right\|_{L_h^2} \right) \\ &\geq -C \left(1 + \left\| \langle x \rangle^N D_+^n u_{j+1} \right\|_{L_h^2}^2 \right) \end{aligned}$$

$$\underline{\text{Estimate for Term}} \left(\langle x \rangle^{2N} D_+^n ((u_1)_j (u_2)_j (u_1)_{j+1}), D_+^n (u_1)_{j+1} \right)_{L_h^2} :$$

For simplicity we will ignore the shifts E. By the product rule we obtain,

$$\begin{aligned} & \left(\langle x \rangle^{2N} D_+^n ((u_1)_j (u_2)_j (u_1)_{j+1}), D_+^n (u_1)_{j+1} \right)_{L_h^2} \\ &= \sum_{i_1+i_2+i_3=n} c_{i_1,i_2,i_3} \left(\langle x \rangle^{2N} D_+^{i_1} (u_1)_j D_+^{i_2} (u_2)_j D_+^{i_3} (u_1)_{j+1}, D_+^n (u_1)_{j+1} \right)_{L_h^2} \end{aligned}$$

For each term we have the estimate,

$$\begin{aligned}
& \left(\langle x \rangle^N D_+^{i_1}(u_1)_j D_+^{i_2}(u_2)_j D_+^{i_3}(u_1)_{j+1}, \langle x \rangle^N D_+^n(u_1)_{j+1} \right)_{L_h^2} \\
& \geq -C \|D_+^{i_1}(u_1)_j\|_\infty \|D_+^{i_2}(u_2)_j\|_\infty \|\langle x \rangle^N D_+^{i_3}(u_1)_{j+1}\|_{L_h^2} \|\langle x \rangle^N D_+^n(u_1)_{j+1}\|_{L_h^2} \\
& \geq -C \left(\|\langle x \rangle^N (u_1)_{j+1}\|_{L_h^2} + \|\langle x \rangle^N D_+^n(u_1)_{j+1}\|_{L_h^2} \right) \|\langle x \rangle^N D_+^n(u_1)_{j+1}\|_{L_h^2} \\
& \geq -C \left(1 + \|\langle x \rangle^N D_+^n u_{j+1}\|_{L_h^2}^2 \right)
\end{aligned}$$

This concludes the proof of the lemma. \square

Corollary 3.4 *Let $u_0 = ((u_1)_0, (u_2)_0)$ where $(u_l)_0 \in S^{-\infty}(\mathbb{R})$ for $l = 1, 2$ and let $K, T, \epsilon > 0$ all be given as in lemma 2.6 and let $N, n \in \mathbb{N}$. Then there exists $0 < k_0 \leq \epsilon$ depending on N and n and there exists $C_{N,n} > 0$ such that if $h \in (0, 1)$, $k \in (0, k_0)$, and $u(x_n, t_j) = (u_1(x_n, t_j), u_2(x_n, t_j))$ is the solution of the difference scheme (2.6) defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ with initial condition u_0 then we have $\|D_+^n(x^N u_j)\|_{L_h^2} < C_{N,n}$ for $t_j \in [0, T]$. Moreover, the constant $C_{N,n}$ is independent of the choice of h and k .*

Proof of Corollary 3.4 By the product rule for D_+ this statement can be proven by induction on n . We shall omit the necessary details here. \square

3.2 Boundedness of Time-Differentiated Extended Discrete Solutions

In this section we will show that a certain time-extension of the discrete solution with domain $\mathbb{R}_h \times \mathbb{R}_k$ remains bounded in the Schwartz semi-norms $\|D_{t,+}^m D_+^n(x^N \cdot)\|_{L_h^2}$ for $n, N \in \mathbb{N}$ and $m \in \{0, 1, 2, 3\}$.

Suppose $u(x_n, t_j) = (u_1(x_n, t_j), u_2(x_n, t_j))$ is mesh function defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ for some $h, k, T > 0$ where $h \in (0, 1)$ and $k \leq T/3$. We shall define an extension of u to $\mathbb{R}_h \times \mathbb{R}_k$ by the following:

Let $\phi(t) \in C_c^\infty(\mathbb{R})$ such that $\phi(t) = 1$ for $t \in [-1, T+1]$ and $\phi(t) = 0$ for $t \notin [-2, T+2]$.

For $t_j > T$ we define recursively

$$u(x_n, t_j) := u(x_n, t_{j-1}) + k D_{t,+} u(x_n, t_{j-2}) + k^2 D_{t,+}^2 u(x_n, t_{j-3})$$

and similarly for $t_j < 0$ we define

$$u(x_n, t_j) := u(x_n, t_{j+1}) - k D_{t,+} u(x_n, t_{j+1}) + k^2 D_{t,+}^2 u(x_n, t_{j+1})$$

Then we define $\hat{u} := \phi \cdot u$, which is clearly an extension of u to $\mathbb{R}_h \times \mathbb{R}_k$ and is compactly supported in time.

It is possible to prove that a finite time discrete solution extended in this way remains bounded in the discrete Schwartz semi-norms $\|D_{t,+}^m D_+^n (x^N u)\|_{L_h^2}$ for $m = 0, 1, 2, 3$ (see lemma 3.5 below). The proof is omitted here but the same statement appears in [9] and is proved there in lemma 3.5. It relies primarily on the estimates of the type given in corollary 3.4 above.

Lemma 3.5 *Let $u_0 = ((u_1)_0, (u_2)_0)$ where $(u_l)_0 \in S^{-\infty}(\mathbb{R})$ for $l = 1, 2$ and let $K, T, \epsilon > 0$ all be given as in lemma 2.6 and let $N, n \in \mathbb{N}$, $m \in \{0, 1, 2, 3\}$. Then there exists $0 < k_0 \leq \epsilon$ depending on m, n and N and there exists $C_{m,N,n} > 0$ such that if $h \in (0, 1)$, $k \in (0, k_0)$, and $u(x_n, t_j)$ is the solution of the difference scheme (2.6) defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ with initial condition u_0 then we have*

$$\|D_{t,+}^m D_+^n (x^N \hat{u}(x, t_j))\|_{L_h^2} \leq C \quad (3.4)$$

for each $t_j \in \mathbb{R}_k$.

4 Obtaining Smooth Solutions from Discrete Solutions

4.1 The Smoothing Operator I_h

We will denote by $L^2(\mathbb{R} \rightarrow \mathbb{R}^n)$ for $n \in \mathbb{N}$ (or simply L^2 where the dimension of the image is understood by context) to be the space of square integrable functions defined on \mathbb{R} with its usual inner product and norm denoted by $(\cdot, \cdot)_{L^2}$ and $\|\cdot\|_{L^2}$ respectively (in contrast to L_h^2 which is the space of square summable functions defined on the mesh \mathbb{R}_h and whose norm is denoted by $\|\cdot\|_{L_h^2}$). Clearly if we restrict a continuum function $u \in L^2$ to \mathbb{R}_h then we may consider it also as a mesh function in L_h^2 .

Lemma 4.1 and corollary 4.2 are based on similar statements found in [3, 18]. They are the key ingredients allowing us to pass from a discrete function to a continuum function while preserving the necessary estimates for our solution (i.e. boundedness of Schwartz semi-norms). The proofs of lemma 4.1 and corollary 4.2 can be found in [3] where the author uses ideas from [18].

Lemma 4.1 *For any $h > 0$ there exists a linear isometry $I_h : L_h^2(\mathbb{R}_h \rightarrow \mathbb{R}^n) \rightarrow L^2(\mathbb{R} \rightarrow \mathbb{R}^n)$ such that if $u \in L_h^2$ then $U := I_h u$ has the following properties:*

1. $U \in C^\infty(\mathbb{R} \rightarrow \mathbb{R}^n)$ (hence we can think of I_h as a "smoothing operator").

2. For any point $x_l \in \mathbb{R}_h$ we have that $U(x_l) = u(x_l)$.

3. For each $j > 0$ the following inequalities hold:

$$\left(\frac{2}{\pi}\right)^j \left\| \frac{\partial^j}{\partial x^j} U \right\|_{L^2} \leq \left\| D_+^j u \right\|_{L_h^2} \leq \left\| \frac{\partial^j}{\partial x^j} U \right\|_{L^2}$$

An explicit formula for $U(x)$ is given by

$$U(x) = \sum_{l=-\infty}^{\infty} u(x_l) \left(\frac{\sin \frac{\pi}{h}(x_l - x)}{\frac{\pi}{h}(x_l - x)} \right) \quad (4.1)$$

Corollary 4.2 Let $M \geq 2$ be an integer and let $h > 0$ be a real number. Suppose u is a mesh function on \mathbb{R}_h such that $x_l^N u(x_l) \in L_h^2 (\mathbb{R}_h \rightarrow \mathbb{R}^n)$ for each $0 \leq N \leq M$. Then for each $j \in \mathbb{N}$, and $0 \leq N \leq M - 2$ we have,

$$\left(\frac{2}{\pi}\right)^j \left\| \frac{\partial^j}{\partial x^j} (x^N U) \right\|_{L^2} \leq \left\| D_+^j (x^N u) \right\|_{L_h^2} \leq \left\| \frac{\partial^j}{\partial x^j} (x^N U) \right\|_{L^2}$$

4.2 Schwartz Boundedness of Smoothly Continued Discrete Solutions

In this section we will show that a certain smooth continuation of the discrete solution remains bounded in the Schwartz semi-norms $\left\| \langle \cdot \rangle^N \partial_x^n \partial_t^m \cdot \right\|_{L_t^\infty L_x^\infty}$.

Suppose $u : \mathbb{R}_h \times \mathbb{R}_k \rightarrow \mathbb{R}^n$ is a mesh function for some $h, k > 0$ which is compactly supported in time for $t_j \in [T_0, T_1]$ and which satisfies the property that there is some $C > 0$ such that $\|u(\cdot, t_j)\|_{L_h^2} \leq C$ for each $t_j \in \mathbb{R}_k$. We will define a smooth continuation of u by the following:

Since u is compactly supported in time we know that for each $x_l \in \mathbb{R}_h$ we have $u(x_l, \cdot) \in L_k^2$, therefore by lemma 4.1 we may apply the operator I_k to $u(x_l, \cdot)$ in t to obtain by (4.1) that for any $t \in \mathbb{R}$,

$$(I_k u)(x_l, t) = \sum_{T_0 \leq t_j \leq T_1} u(x_l, t_j) \left[\frac{\sin \frac{\pi}{k}(t - t_j)}{\frac{\pi}{k}(t - t_j)} \right]$$

and therefore,

$$\begin{aligned} \|I_k u(\cdot, t)\|_{L_h^2} &\leq \sum_{T_0 \leq t_j \leq T_1} \left| \frac{\sin \frac{\pi}{k}(t - t_j)}{\frac{\pi}{k}(t - t_j)} \right| \cdot \|u(\cdot, t_j)\|_{L_h^2} \\ &\leq \sum_{T_0 \leq t_j \leq T_1} \left\| \frac{\sin y}{y} \right\|_\infty \cdot \|u(\cdot, t_j)\|_{L_h^2} < \infty \end{aligned} \quad (4.2)$$

Hence by lemma 4.1 we can apply the smoothing operator I_h to $I_k u(\cdot, t)$ in the x variable for each $t \in \mathbb{R}$ to obtain a continuum function $Iu := I_h I_k u$. By linearity of I_h it follows that Iu is given by,

$$Iu(x, t) = \sum_{T_0 \leq t_j \leq T_1} I_h u(x, t_j) \left[\frac{\sin \frac{\pi}{k}(t - t_j)}{\frac{\pi}{k}(t - t_j)} \right]$$

and since for each t_j we have $\frac{\sin \frac{\pi}{k}(t - t_j)}{\frac{\pi}{k}(t - t_j)}$ is smooth in t and also for each j the function $I_h u(x, t_j)$ is smooth in x we see that $Iu \in C^\infty(\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^n)$ and by lemma 4.1 $Iu(x_l, t_j) = u(x_l, t_j)$ for any $(x_l, t_j) \in \mathbb{R}_h \times \mathbb{R}_k$. Moreover, it is clear from the above formula that for any $m \in \mathbb{N}$ we have $\partial_t^m Iu = I_h (\partial_t^m I_k u)$. Given a discrete solution u from lemma 2.6 we may now construct a smooth continuation $I\hat{u}$ and prove that it remains bounded in the continuum Schwartz seminorms - this is the statement of lemma 4.3 below. Its proof is the same as the proof of lemma 4.3 in [9] so we will omit it here.

If $u(x, t) = (u_1(x, t), u_2(x, t)) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2$ is any function then we will use the notations $\|u(\cdot, t)\|_{L_x^\infty} := \|u_1(\cdot, t)\|_\infty + \|u_2(\cdot, t)\|_\infty$ and $\|u\|_{L_t^\infty L_x^\infty} := \|\|u\|_{L_x^\infty}\|_{L_t^\infty}$.

Lemma 4.3 *Let $u_0 = ((u_1)_0, (u_2)_0)$ where $(u_l)_0 \in S^{-\infty}(\mathbb{R})$ for $l = 1, 2$ and let $K, T, \epsilon > 0$ all be given as in lemma 2.6 and let $N, n \in \mathbb{N}$, $m \in \{0, 1, 2\}$. Then there exists $0 < k_0 \leq \epsilon$ and $C > 0$ both depending on m, n and N such that if $h \in (0, 1)$, $k \in (0, k_0)$, and $u(x_n, t_j) = (u_1(x_n, t_j), u_2(x_n, t_j))$ is the solution of the difference scheme (2.6) defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ with initial condition u_0 then we have*

$$\|\langle \cdot \rangle^N \partial_x^n \partial_t^m I\hat{u}\|_{L_t^\infty L_x^\infty} \leq C$$

4.3 Proof of Local Existence for the Generalized NLS equation in $S^{-\infty}$

By using corollary 4.2 and the Arzela-Ascoli theorem we shall now construct a smooth solution to (1.2) lying in $S^{-\infty}(\mathbb{R} \times [0, T])$ that comes from the discrete solution constructed in lemma 2.6. Theorem 1.1 and its proof are completely analogous to the corresponding results given by Bondareva for the KdV equation (see [3] theorem 2) and by the author for the modified KdV equation (see [9] theorem 1.1).

Proof of Theorem 1.1 (existence)

Since $u_0 \in S^{-\infty}(\mathbb{R})$ it follows that there is some $K > 0$ such that for any $0 < h < 1$ we have $\|u_0\|_{S_h} \leq K$. Therefore, by lemma 2.6, there exists $T, L, \epsilon > 0$ such that if $h \in (0, 1)$ and

$k \in (0, \epsilon)$ then there is a solution to the difference scheme (2.6) with initial condition u_0 defined on $\mathbb{R}_h \times (\mathbb{R}_k \cap [0, T])$ and we denote this solution by $u^{h,k}$. Let $U^{h,k} = (U_1^{h,k}, U_2^{h,k}) := I\hat{u}^{h,k}$. From lemma 4.3 we know that for every $N, n \in \mathbb{N}$, $m \in \{0, 1, 2\}$ there exists $0 < k_0(m, n, N) \leq \epsilon$ and there exists $C_{m,N,n} > 0$ such that if $h \in (0, 1)$, $k \in (0, k_0)$, then we have

$$\left\| \langle \cdot \rangle^N \partial_x^n \partial_t^m U^{h,k} \right\|_{L_t^\infty L_x^\infty} \leq C \quad (4.3)$$

From the family of functions $\{U^{h,k}\}_{h \in (0,1), k \in (0, k_0(1,1,0))}$ we now wish to extract a convergent subsequence by using the Arzela-Ascoli theorem (this theorem can be found for example in [16]).

Let $(x_0, t_0), (x_1, t_1)$ be points in $\mathbb{R} \times \mathbb{R}$. By lemma 4.3 and by the intermediate value theorem we have

$$\begin{aligned} |U^{h,k}(x_0, t_0) - U^{h,k}(x_1, t_1)| &\leq |U^{h,k}(x_0, t_0) - U^{h,k}(x_0, t_1)| + |U^{h,k}(x_0, t_1) - U^{h,k}(x_1, t_1)| \\ &\leq C \left(|\partial_t U_1^{h,k}(x_0, \tilde{t})| + |\partial_t U_2^{h,k}(x_0, \tilde{t})| \right) \cdot |t_0 - t_1| \\ &\quad + C \left(|\partial_x U_1^{h,k}(\tilde{x}, t_1)| + |\partial_x U_2^{h,k}(\tilde{x}, t_1)| \right) \cdot |x_0 - x_1| \\ &\leq \|\partial_t U^{h,k}\|_{L_t^\infty L_x^\infty} \cdot |t_0 - t_1| + \|\partial_x U^{h,k}\|_{L_t^\infty L_x^\infty} \cdot |x_0 - x_1| \\ &\leq C |(x_0, t_0) - (x_1, t_1)| \end{aligned} \quad (4.4)$$

which shows that the family of functions $U^{h,k}$ is equicontinuous on $\mathbb{R} \times \mathbb{R}$.

From (4.3) it follows that the family $U^{h,k}$ is also bounded uniformly for $h \in (0, 1)$, $k \in (0, k_0(1, 1, 0))$. Hence, by the Arzela-Ascoli theorem we may construct a subsequence U^{h_i, k_i} , where of course $h_i, k_i \searrow 0$ as $i \rightarrow \infty$, converging uniformly on compact sets to a function $U \in C^0(\mathbb{R} \times \mathbb{R})$.

The above argument can also be made for the family of functions $\partial_x U^{h_i, k_i}$ for $h \in (0, 1)$, $k \in (0, k_0(1, 2, 0))$. Namely, estimate (4.3) implies that the family is bounded uniformly and also that we may use estimate (4.4) with $U^{h,k}$ replaced by $\partial_x U^{h_i, k_i}$ to see that it is also an equicontinuous family. Thus we conclude that there is some $V \in C^0(\mathbb{R} \times \mathbb{R})$ and a subsequence $\partial_x U^{h_i, k_i}$ converging uniformly on compact sets to V . Since we have uniform convergence on compact sets for U^{h_i, k_i} and $\partial_x U^{h_i, k_i}$ it follows that U is differentiable in x and $\partial_x U = V$ on $\mathbb{R} \times \mathbb{R}$.

By repeating the same argument we conclude by induction that for each $p \in \mathbb{N}$ the function $\partial_x^{p-1} U \in C^0(\mathbb{R} \times \mathbb{R})$ is differentiable in x because the sequence $\partial_x^p U^{h_i, k_i}$ for $h \in (0, 1)$ and $k \in (0, k_0(1, p+1, 0))$ is bounded uniformly by (4.3) and is equicontinuous by (4.4) and hence it

has a subsequence uniformly convergent on compact subsets of $\mathbb{R} \times \mathbb{R}$ to $\partial_x \partial_x^{p-1} U$. In this way we will obtain a countable array of subsequences, one for each $p \in \mathbb{N}$ and from this array we extract a diagonal subsequence. From this diagonal subsequence it will follow that for each $p \in \mathbb{N}$ we have $\partial_x^p U^{h_l, k_l} \rightarrow \partial_p U$ uniformly on compact sets.

Consider the family of functions $\partial_t U^{h_l, k_l}$ for $h \in (0, 1)$ and $k \in (0, k_0(2, 1, 0))$. Estimate (4.3) for $m = 1$ implies that the family is bounded uniformly and also that we may use estimate (4.4) with $U^{h, k}$ replaced by $\partial_t U^{h_l, k_l}$ to see that it is also an equicontinuous family. Hence we may, as before for x , conclude that U is differentiable in t and construct a subsequence of $\partial_t U^{h_l, k_l}$ uniformly convergent on compact sets to $\partial_t U$. From this subsequence of (h_l, k_l) we consider the family $\partial_x \partial_t U^{h_l, k_l}$ for $h \in (0, 1)$ and $k \in (0, k_0(2, 2, 0))$. Again estimate (4.3) implies that the family is bounded uniformly and also we may use estimate (4.4) with $U^{h, k}$ replaced by $\partial_x \partial_t U^{h_l, k_l}$ to see that it is also an equicontinuous family. Thus we may again extract a subsequence U^{h_l, k_l} to see that $\partial_t U$ is differentiable in x and $\partial_x \partial_t U^{h_l, k_l} \rightarrow \partial_x \partial_t U$ uniformly on compact sets. Continuing inductively we consider the sequence of functions $\partial_x^p \partial_t U^{h_l, k_l}$ for $h \in (0, 1)$, $k \in (0, k_0(2, p+1, 0))$. It is equicontinuous by (4.4) and from (4.3) it is uniformly bounded, thus we conclude that $\partial_x^{p-1} \partial_t U$ is differentiable in x and we may extract a subsequence so that $\partial_x^p \partial_t U^{h_l, k_l} \rightarrow \partial_x^p \partial_t U$.

Continuing in this way we will again obtain an array of subsequences of U^{h_l, k_l} , one for each $p \in \mathbb{N}$. By taking a diagonal subsequence we obtain a subsequence such that for each $p \in \mathbb{N}$ and for $q = 0, 1$ we have $\partial_x^p \partial_t^q U^{h_l, k_l} \rightarrow \partial_x^p \partial_t^q U$ uniformly on compact subsets of $\mathbb{R} \times \mathbb{R}$. In addition, it follows that for any $N \in \mathbb{N}$ the sequence $\langle x \rangle^N \partial_x^p \partial_t^q U^{h_l, k_l} \rightarrow \langle x \rangle^N \partial_x^p \partial_t^q U$ uniformly on compact subsets of $\mathbb{R} \times \mathbb{R}$ because for any compact set $X \subset \mathbb{R} \times \mathbb{R}$ we have the inequality

$$\left| \langle x \rangle^N \partial_x^p \partial_t^q U^{h, k}(x, t) - \langle x \rangle^N \partial_x^p \partial_t^q U(x, t) \right| \leq \max_{x \in X} \langle x \rangle^N \cdot |\partial_x^p \partial_t^q U^{h, k}(x, t) - \partial_x^p \partial_t^q U(x, t)|$$

By construction we can see that U satisfies the following conditions:

1. $\partial_x^p U$ exists for each $p \in \mathbb{N}$ and is continuous (i.e. it lies in $C^0(\mathbb{R} \times \mathbb{R})$).
2. $\partial_x^q \partial_t^p U$ exists for each $p, q \in \mathbb{N}$ and is continuous (i.e. it lies in $C^0(\mathbb{R} \times \mathbb{R})$).
3. If $p + q = p' + q'$ then $\partial_x^q \partial_t^p U = \partial_x^{q'} \partial_t^{p'} U$.

We now claim that the complex-valued function $U_1 + iU_2$ is a solution to (1.2) (and we shall also

identify this solution by U). To prove it we fix a point $(x, t) \in \mathbb{R} \times [0, T]$ and show that the system (2.1), (2.2) is satisfied by its components U_1 and U_2 at (x, t) . From our final subsequence of pairs (h_l, k_l) above we first construct points $(x_l, t_l) \in \mathbb{R}_{h_l} \times (\mathbb{R}_{k_l} \cap [0, T])$ to be the nearest points in the grid to (x, t) (note: in this context $x_l \neq l \cdot h_l$ and $t_l \neq l \cdot k_l$). It then follows that $(x_l, t_l) \rightarrow (x, t)$ as $l \rightarrow \infty$. By construction we have for each $l \in \mathbb{N}$ that $(U_1^{h_l, k_l}, U_2^{h_l, k_l})$ satisfies the pair of difference equations (2.3), (2.4) at point (x_l, t_l) . Replace the discrete derivatives in t and x of equations (2.3) and (2.4) by ordinary derivatives at intermediate points $(\tilde{x}_l, \tilde{t}_l)$ (possibly different intermediate points for each term containing derivatives). We will then obtain a sum of products of terms of the form $\partial_t U_r^{h_l, k_l}(x_l, \tilde{t}_l)$, $U_r^{h_l, k_l}(x_l, t_l)$, $U_r^{h_l, k_l}(x_l, t_l + k_l)$, $\partial_x^2 U_r^{h_l, k_l}(\tilde{x}_l, t_l)$, $f_r(x_l, t_l)$, and $g_r(x_l, t_l)$ for $r = 1, 2$. By continuity of f and g we see that $f(x_l, t_l) \rightarrow f(x, t)$ and $g(x_l, t_l) \rightarrow g(x, t)$ as $l \rightarrow \infty$. Moreover, since $(U_1^{h_l, k_l}(x, t), U_2^{h_l, k_l}(x, t)) \rightarrow (U_1(x, t), U_2(x, t))$ as $l \rightarrow \infty$ and

$$\begin{aligned} & \left| (U_1^{h_l, k_l}(x_l, t_l), U_2^{h_l, k_l}(x_l, t_l)) - (U_1(x, t), U_2(x, t)) \right| \\ & \leq \left| (U_1^{h_l, k_l}(x_l, t_l), U_2^{h_l, k_l}(x_l, t_l)) - (U_1^{h_l, k_l}(x, t), U_2^{h_l, k_l}(x, t)) \right| \\ & \quad + \left| (U_1^{h_l, k_l}(x, t), U_2^{h_l, k_l}(x, t)) - (U_1(x, t), U_2(x, t)) \right| \end{aligned}$$

it follows by equicontinuity of the family $(U_1^{h_l, k_l}, U_2^{h_l, k_l})$ for h, k sufficiently small that $(U_1^{h_l, k_l}, U_2^{h_l, k_l}) \rightarrow (U_1(x, t), U_2(x, t))$. We may use the same convergence argument for the other terms in the equation to show that as $l \rightarrow \infty$ the equations (2.3), (2.4) becomes equations (2.1), (2.2) at the point (x, t) . Therefore (U_1, U_2) satisfies the equations (2.1), (2.2) and hence $U = U_1 + iU_2$ satisfies (1.2).

Since U satisfies the equation (1.2) it follows that $\partial_t U$ is also differentiable in time and its higher time derivatives can be written in terms of the lower x derivatives. The derivatives also clearly commute as was mentioned above in condition 3, therefore it follows that $U \in C^\infty(\mathbb{R} \times [0, T] \rightarrow \mathbb{C})$.

Furthermore, we can show that the limit function U is in $S^{-\infty}(\mathbb{R} \times [0, T] \rightarrow \mathbb{C})$. By taking the limit of $\langle \cdot \rangle^N \partial_x^n \partial_t^m U^{h_l, k_l}(x, t)$ as $l \rightarrow \infty$ we can see that (4.3) also holds for the function U with $n, N \in \mathbb{N}$ and $m = 0, 1$. By repeatedly using the equation (1.2) we may write $\langle \cdot \rangle^N \partial_x^n \partial_t^m U$ as a sum of products of terms of the form $\langle \cdot \rangle^N \partial_x^n U$ each of which can be bounded by some constant depending on $N, n \in \mathbb{N}$ by using the limiting case of (4.3) and this implies that for any $m, n, N \in \mathbb{N}$ we have,

$$\left\| \langle \cdot \rangle^N \partial_x^n \partial_t^m U \right\|_{L_t^\infty L_x^\infty} \leq C_{m, n, N}$$

which shows that $U \in S^{-\infty}(\mathbb{R} \times [0, T] \rightarrow \mathbb{C})$. \square

4.4 Proof of Local Existence for the NLS equation in S^β when $\beta \leq 0$

Now we shall now construct smooth solutions to (1.1) lying in $S^\beta(\mathbb{R} \times [0, T] \rightarrow \mathbb{C})$ for $\beta \leq 0$ that come from adding an above solution of (1.2) to the asymptotic solution constructed in lemma A.2.

Proof of Theorem 1.2 (existence)

By lemma A.2 there exists an asymptotic solution $f(x, t) \in S^\beta(\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C})$ of the initial value problem (1.1) whose expansion coefficients satisfy the desired property. Let $u_0(x) = w_0(x) - f(x, 0)$ and let $g := if_t + f_{xx} + \mu \bar{f} f^2$. By construction $u_0 \in S^{-\infty}(\mathbb{R} \rightarrow \mathbb{C})$. Moreover f and g satisfy the hypotheses of theorem 1.1. Therefore there exists a $T > 0$ and a solution $u(x, t) \in S^{-\infty}(\mathbb{R} \times [0, T])$ to equation (1.2). Let $w(x, t) := u(x, t) + f(x, t)$. Since u satisfies (1.2) it follows that w satisfies (1.1). Moreover, since $u \sim 0$ it follows that w and f have the same asymptotic expansions and in particular the coefficients in the asymptotic expansions of w satisfy the second statement of the theorem. Finally, since $f(x, t) \in S^\beta(\mathbb{R} \times [0, T] \rightarrow \mathbb{C})$ it follows that $w(x, t) \in S^\beta(\mathbb{R} \times [0, T] \rightarrow \mathbb{C})$. \square

5 Uniqueness of Solutions

5.1 Uniqueness in $S^{-\infty}$ for the Generalized NLS Equation

In this section we shall prove uniqueness of solutions in $S^{-\infty}(\mathbb{R} \times [0, T])$ for (1.2) by using Gronwall's Inequality. We shall state this inequality below and we refer the reader to [9] for a short proof.

Lemma 5.1 *Let $T > 0$ and $c_1, c_2 \in \mathbb{R}$ be given and $c_1 \neq 0$. Suppose $\eta : [0, T] \rightarrow \mathbb{R}$ is a nonnegative, differentiable function and that for each $t \in [0, T]$ we have*

$$\frac{d\eta}{dt}(t) \leq c_1 \eta(t) + c_2$$

Then for each $t \in [0, T]$ we have

$$\eta(t) \leq e^{c_1 t} \left(\eta(0) - \frac{c_1}{c_2} \right) - \frac{c_1}{c_2}$$

Proof of Theorem 1.1 (uniqueness)

Suppose $u(x, t) = u_1(x, t) + iu_2(x, t), v(x, t) = v_1(x, t) + iv_2(x, t) \in S^{-\infty}(\mathbb{R} \times [0, T] \rightarrow \mathbb{R})$ are two

solutions of (1.2) with initial data $u_0 = ((u_1)_0, (u_2)_0) \in S^{-\infty}(\mathbb{R} \rightarrow \mathbb{C})$. Then by equations (rgNLS) and (igNLS) we have

$$\begin{aligned}(u_1)_t + (u_2)_{xx} + \mu(u_1^2 u_2 + u_2^3 + 2u_1 u_2 f_1 + u_1^2 f_2 + 3u_2^2 f_2 + 2f_1 f_2 u_1 + u_2 f_1^2 + 3u_2 f_2^2) + g_2 &= 0 \\ (v_1)_t + (v_2)_{xx} + \mu(v_1^2 v_2 + v_2^3 + 2v_1 v_2 f_1 + v_1^2 f_2 + 3v_2^2 f_2 + 2f_1 f_2 v_1 + v_2 f_1^2 + 3v_2 f_2^2) + g_2 &= 0\end{aligned}$$

and

$$\begin{aligned}(u_2)_t - (u_1)_{xx} - \mu(u_1^3 + u_1 u_2^2 + u_1 f_2^2 + 3u_1^2 f_1 + u_2^2 f_1 + 2u_1 u_2 f_2 + 3u_1 f_1^2 + 2f_1 f_2 u_2) - g_1 &= 0 \\ (v_2)_t - (v_1)_{xx} - \mu(v_1^3 + v_1 v_2^2 + v_1 f_2^2 + 3v_1^2 f_1 + v_2^2 f_1 + 2v_1 v_2 f_2 + 3v_1 f_1^2 + 2f_1 f_2 v_2) - g_1 &= 0\end{aligned}$$

Let $q(x, t) = (q_1(x, t), q_2(x, t)) := u(x, t) - v(x, t)$. By subtracting the above two equations we see that the components of q satisfy the equations

$$\begin{aligned}(q_1)_t + (q_2)_{xx} + \mu[u_1 u_2 q_1 + u_1 v_1 q_2 + v_1 v_2 q_1 + q_2(u_2^2 + u_2 v_2 + v_2^2) \\ + 2f_1 u_2 q_1 + 2f_1 v_1 q_2 + f_2 q_1(u_1 + v_1) + 3f_2 q_2(u_2 + v_2) + 2f_1 f_2 q_1 + f_1^2 q_2 + 3f_2^2 q_2] &= 0 \\ (q_2)_t - (q_1)_{xx} - \mu[u_1 u_2 q_2 + u_1 v_2 q_2 + v_2^2 q_1 + q_1(u_1^2 + u_1 v_1 + v_1^2) \\ + f_2^2 q_1 + 3f_1 q_1(u_1 + v_1) + f_2 q_2(u_2 + v_2) + 3f_1^2 q_1 + 2f_1 f_2 q_2 + 2f_2 u_2 q_1 + 2f_1 v_1 q_2] &= 0\end{aligned}$$

Thus if we multiply the first equation by q_1 and the second equation by q_2 then add them together and integrate by parts in x over $(-\infty, \infty)$ we get an estimate of the form

$$\frac{d}{dt} \int_{\mathbb{R}} (q_1^2 + q_2^2) dx \leq C_{u,v,f,T} \int_{\mathbb{R}} (q_1^2 + q_2^2) dx$$

and moreover, $q(\cdot, 0) = 0$, therefore by lemma 5.1 it follows that $\|q(\cdot, t)\|_{L^2} = 0$ for all $t \in [0, T]$ and since q is smooth this implies that $q(x, t) = 0$ for all $(x, t) \in \mathbb{R} \times [0, T]$. \square

5.2 Uniqueness in S^β for the NLS Equation When $\beta \leq 0$

In this section we shall prove uniqueness of solutions in $S^\beta(\mathbb{R} \times [0, T])$ for (1.1) when $\beta \leq 0$. First we will need the following lemma.

Lemma 5.2 *Let $I \subset \mathbb{R}$ be an interval and $\beta \leq 0$. Suppose $w(x, t) \in S^\beta(\mathbb{R} \times I \rightarrow \mathbb{C})$ is a solution to (1.1) with initial data $w_0 \in S^\beta(\mathbb{R} \rightarrow \mathbb{C})$ and that $w(x, t) \sim \sum_{k=0}^{\infty} (a_k^\pm(t) + i b_k^\pm(t)) x^{\beta_k}$ as $x \rightarrow \pm\infty$. Then $\sum_{k=0}^{\infty} (a_k^\pm(t) + i b_k^\pm(t)) x^{\beta_k}$ is a formal solution to (1.1).*

Proof of Lemma 5.2 By symmetry it suffices to show that the positive x asymptotic expansion satisfies equation (1.1). Let $A_0 = \{\beta_j\}_{j=0}^\infty$ and let $J \subset I$ be a compact interval. We enlarge A_0 to the set Γ defined in appendix A having the properties mentioned in lemma A.1. Let us re-write the asymptotic expansion as $\sum_{k=0}^\infty (a_k^+(t) + ib_k^+(t)) x^{\gamma_k}$ where $a_k^+(t) + ib_k^+(t) = 0$ if $\gamma_k \notin A_0$. By definition of being asymptotic it follows that for every $N \in \mathbb{N}$ we may write

$$w(x, t) = \sum_{k=0}^N (a_k^+(t) + ib_k^+(t)) x^{\gamma_k} + R_N(x, t)$$

for $x > 1$ and $t \in J$ where $\partial_t^i \partial_x^j R_N(x, t) = O(|x|^{\gamma_{N+1}-j})$ for every $i, j \in \mathbb{N}$. Let

$f_N(x, t) = \sum_{k=0}^N (a_k^+(t) + ib_k^+(t)) x^{\gamma_k}$. Then by plugging w into (1.1) and separating real and imaginary parts, it follows as we see in (A.1) and (A.2) that for some $M \leq N$ we have

$$\begin{aligned} \sum_{j=0}^M \left[\dot{a}_j^+ + \mu \sum_{\substack{l,m,n \\ \gamma_l+\gamma_m+\gamma_n=\gamma_j}} -a_l^+ a_m^+ b_n^+ + a_l^+ a_n^+ b_m^+ + a_n^+ a_m^+ b_l^+ + b_l^+ b_m^+ b_n^+ + \sum_{\substack{p \\ \gamma_p-2=\gamma_j}} b_p^+ \gamma_p (\gamma_p - 1) \right] x^{\gamma_j} \\ + O(|x|^{2\gamma_0+\gamma_{N+1}}) = 0 \end{aligned} \quad (5.1)$$

and

$$\begin{aligned} \sum_{j=0}^M \left[\dot{b}_j^+ - \mu \sum_{\substack{l,m,n \\ \gamma_l+\gamma_m+\gamma_n=\gamma_j}} a_l^+ a_m^+ a_n^+ + a_l^+ b_n^+ b_m^+ + b_n^+ a_m^+ b_l^+ - b_l^+ b_m^+ a_n^+ - \sum_{\substack{p \\ \gamma_p-2=\gamma_j}} a_p^+ \gamma_p (\gamma_p - 1) \right] x^{\gamma_j} \\ + O(|x|^{2\gamma_0+\gamma_{N+1}}) = 0 \end{aligned} \quad (5.2)$$

We may assume that N is sufficiently large so that $M \geq 1$ and $2\gamma_0 + \gamma_{N+1} < \gamma_1$. Since the above equation must hold for all $x > 1$ we may divide by x^{γ_0} to obtain from (5.1) and (5.2) that

$$\begin{aligned} \dot{a}_0^+ + \mu \sum_{\substack{l,m,n \\ \gamma_l+\gamma_m+\gamma_n=\gamma_0}} -a_l^+ a_m^+ b_n^+ + a_l^+ a_n^+ b_m^+ + a_n^+ a_m^+ b_l^+ + b_l^+ b_m^+ b_n^+ \\ + \sum_{\substack{p \\ \gamma_p-2=\gamma_0}} b_p^+ \gamma_p (\gamma_p - 1) + O(|x|^{\gamma_1-\gamma_0}) = 0 \end{aligned}$$

and

$$\begin{aligned} \dot{b}_0^+ - \mu \sum_{\substack{l,m,n \\ \gamma_l + \gamma_m + \gamma_n = \gamma_0}} & a_l^+ a_m^+ a_n^+ + a_l^+ b_n^+ b_m^+ + b_n^+ a_m^+ b_l^+ - b_l^+ b_m^+ a_n^+ \\ & - \sum_{\substack{p \\ \gamma_p - 2 = \gamma_0}} a_p^+ \gamma_p (\gamma_p - 1) + O(|x|^{\gamma_1 - \gamma_0}) = 0 \end{aligned}$$

and hence

$$\dot{a}_0^+ + \mu \sum_{\substack{l,m,n \\ \gamma_l + \gamma_m + \gamma_n = \gamma_0}} -a_l^+ a_m^+ b_n^+ + a_l^+ a_n^+ b_m^+ + a_n^+ a_m^+ b_l^+ + b_l^+ b_m^+ b_n^+ + \sum_{\substack{p \\ \gamma_p - 2 = \gamma_0}} b_p^+ \gamma_p (\gamma_p - 1) = 0$$

and

$$\dot{b}_0^+ - \mu \sum_{\substack{l,m,n \\ \gamma_l + \gamma_m + \gamma_n = \gamma_0}} a_l^+ a_m^+ a_n^+ + a_l^+ b_n^+ b_m^+ + b_n^+ a_m^+ b_l^+ - b_l^+ b_m^+ a_n^+ - \sum_{\substack{p \\ \gamma_p - 2 = \gamma_0}} a_p^+ \gamma_p (\gamma_p - 1) = 0$$

Continuing in the same way we may assume that N is sufficiently large so that $2\gamma_0 + \gamma_{N+1} - 1 < \gamma_2$.

Dividing (5.1) and (5.2) by x^{γ_1} we obtain that

$$\begin{aligned} \dot{a}_1^+ + \mu \sum_{\substack{l,m,n \\ \gamma_l + \gamma_m + \gamma_n = \gamma_1}} & -a_l^+ a_m^+ b_n^+ + a_l^+ a_n^+ b_m^+ + a_n^+ a_m^+ b_l^+ + b_l^+ b_m^+ b_n^+ \\ & + \sum_{\substack{p \\ \gamma_p - 2 = \gamma_1}} b_p^+ \gamma_p (\gamma_p - 1) + O(|x|^{\gamma_2 - \gamma_1}) = 0 \end{aligned}$$

and

$$\begin{aligned} \dot{b}_1^+ - \mu \sum_{\substack{l,m,n \\ \gamma_l + \gamma_m + \gamma_n = \gamma_1}} & a_l^+ a_m^+ a_n^+ + a_l^+ b_n^+ b_m^+ + b_n^+ a_m^+ b_l^+ - b_l^+ b_m^+ a_n^+ \\ & - \sum_{\substack{p \\ \gamma_p - 2 = \gamma_1}} a_p^+ \gamma_p (\gamma_p - 1) + O(|x|^{\gamma_2 - \gamma_1}) = 0 \end{aligned}$$

and hence

$$\dot{a}_1^+ + \mu \sum_{\substack{l,m,n \\ \gamma_l + \gamma_m + \gamma_n = \gamma_1}} -a_l^+ a_m^+ b_n^+ + a_l^+ a_n^+ b_m^+ + a_n^+ a_m^+ b_l^+ + b_l^+ b_m^+ b_n^+ + \sum_{\substack{p \\ \gamma_p - 2 = \gamma_1}} b_p^+ \gamma_p (\gamma_p - 1) = 0$$

and

$$\dot{b}_1^+ - \mu \sum_{\substack{l,m,n \\ \gamma_l + \gamma_m + \gamma_n = \gamma_1}} a_l^+ a_m^+ a_n^+ + a_l^+ b_n^+ b_m^+ + b_n^+ a_m^+ b_l^+ - b_l^+ b_m^+ a_n^+ - \sum_{\substack{p \\ \gamma_p - 2 = \gamma_1}} a_p^+ \gamma_p (\gamma_p - 1) = 0$$

This process may be repeated inductively to obtain from (5.1) and (5.2) that for any $j \in \mathbb{N}$ we have

$$\dot{a}_j^+ + \mu \sum_{\substack{l,m,n \\ \gamma_l + \gamma_m + \gamma_n = \gamma_j}} -a_l^+ a_m^+ b_n^+ + a_l^+ a_n^+ b_m^+ + a_n^+ a_m^+ b_l^+ + b_l^+ b_m^+ b_n^+ + \sum_{\substack{p \\ \gamma_p - 2 = \gamma_j}} b_p^+ \gamma_p (\gamma_p - 1) = 0$$

and

$$\dot{b}_j^+ - \mu \sum_{\substack{l,m,n \\ \gamma_l + \gamma_m + \gamma_n = \gamma_j}} a_l^+ a_m^+ a_n^+ + a_l^+ b_n^+ b_m^+ + b_n^+ a_m^+ b_l^+ - b_l^+ b_m^+ a_n^+ - \sum_{\substack{p \\ \gamma_p - 2 = \gamma_j}} a_p^+ \gamma_p (\gamma_p - 1) = 0$$

and hence $\sum_{k=0}^{\infty} (a_k^+(t) + i b_k^+(t)) x^{\gamma_k}$ is a formal solution to (1.1). \square

Proof of Theorem 1.2 (uniqueness)

Suppose $w(x, t), r(x, t) \in S^\beta(\mathbb{R} \times [0, T] \rightarrow \mathbb{C})$ are two solutions of (1.1) with initial data $w_0(x) \in S^\beta(\mathbb{R} \rightarrow \mathbb{C})$ and that

$$w_0(x) \sim \sum_{k=0}^{\infty} (p_k^\pm + iq_k^\pm) x^{\beta_k} \quad w(x, t) \sim \sum_{k=0}^{\infty} (a_k^\pm(t) + ib_k^\pm(t)) x^{\alpha_k} \quad r(x, t) \sim \sum_{k=0}^{\infty} (c_k^\pm(t) + id_k^\pm(t)) x^{\delta_k}$$

as $x \rightarrow \pm\infty$. Let $B_0 = \{\beta_k\}_{k=0}^{\infty}$, $A_0 = \{\alpha_k\}_{k=0}^{\infty}$, and $D_0 = \{\delta_k\}_{k=0}^{\infty}$. By lemma 5.2 $\sum_{k=0}^{\infty} (a_k^\pm(t) + ib_k^\pm(t)) x^{\alpha_k}$ and $\sum_{k=0}^{\infty} (c_k^\pm(t) + id_k^\pm(t)) x^{\delta_k}$ are formal solutions to (1.1) with initial data $\sum_{k=0}^{\infty} (p_k^\pm + iq_k^\pm) x^{\beta_k}$ and hence we may assume that $B_0 \subset A_0$ and $B_0 \subset D_0$. Let $\Lambda = A_0 \cup D_0$ and $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be the set constructed in appendix A from Λ having the properties stated in lemma A.1. Then after reindexing we may rewrite the asymptotic expansions for w_0 , $w(x, t)$, and $r(x, t)$ as

$$w_0(x) \sim \sum_{k=0}^{\infty} (p_k^\pm + iq_k^\pm) x^{\gamma_k} \quad w(x, t) \sim \sum_{k=0}^{\infty} (a_k^\pm(t) + ib_k^\pm(t)) x^{\gamma_k} \quad r(x, t) \sim \sum_{k=0}^{\infty} (c_k^\pm(t) + id_k^\pm(t)) x^{\gamma_k}$$

where $p_k^\pm + iq_k^\pm = 0$ if $\gamma_k \notin B_0$, $a_k^\pm(t) + ib_k^\pm(t) = 0$ if $\gamma_k \notin A_0$, and $c_k^\pm(t) + id_k^\pm(t) = 0$ if $\gamma_k \notin D_0$. Since $\sum_{k=0}^{\infty} (a_k^\pm(t) + ib_k^\pm(t)) x^{\gamma_k}$ and $\sum_{k=0}^{\infty} (c_k^\pm(t) + id_k^\pm(t)) x^{\gamma_k}$ are formal solutions with initial data $\sum_{k=0}^{\infty} (p_k^\pm + iq_k^\pm) x^{\gamma_k}$ it follows that the coefficient pairs (a_k^\pm, b_k^\pm) and (c_k^\pm, d_k^\pm) both satisfy equa-

tions (A.1) and (A.2) with the same initial data. Hence for all $k \in \mathbb{N}$ and all $t \in [0, T]$ we have $a_k^\pm(t) + ib_k^\pm(t) = c_k^\pm(t) + id_k^\pm(t)$ so that $w(x, t) - r(x, t) \in S^{-\infty}(\mathbb{R} \times [0, T] \rightarrow \mathbb{C})$.

Let $u(x, t) = w(x, t) - r(x, t)$. Then $u(x, t)$ satisfies (1.2) with initial condition $u_0(x) = 0$ and where $f(x, t) = r(x, t)$ and $g(x, t) = 0$. By uniqueness of solutions to (1.2) in $S^{-\infty}(\mathbb{R} \times [0, T])$, which was proven in theorem 1.1, it follows that $u(x, t) = 0$ for all $(x, t) \in \mathbb{R} \times [0, T]$. \square

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A Appendix: Existence of an Asymptotic Solution

In this section we will prove existence of an asymptotic solution to (1.1). First we will need the following lemma.

Let $A_0 = \{\beta_j\}_{j=0}^\infty$. Where $0 \geq \beta_0 > \beta_1 > \dots$, and $\lim_{j \rightarrow \infty} \beta_j = -\infty$. We enlarge A_0 to the set Γ given by,

$$\Gamma := \left\{ \sum_{p=1}^k \beta_{i_p} - 2l : k \geq 1, l \geq 0, k, l \in \mathbb{Z}, \beta_{i_p} \in A_0 \right\}$$

Lemma A.1 *The set Γ has the following properties:*

1. $A_0 \subset \Gamma$
2. Γ is countable.
3. Γ is bounded above by β_0 .
4. If $\gamma_l, \gamma_m, \gamma_n$ are all in Γ then $\gamma_l + \gamma_m + \gamma_n$ is in Γ .
5. If γ_p is in Γ then $\gamma_p - 2$ is also in Γ .
6. Γ is lower finite, i.e. $\Gamma \cap [-M, \infty)$ is finite for every $M > 0$.

Proof of Lemma A.1 Statements 1 to 5 follow easily from the definition of Γ so we shall only prove lower finiteness here.

Let $\Sigma A := \left\{ \sum_{p=1}^k \beta_{i_p} : k \geq 1, k \in \mathbb{Z}, \beta_{i_p} \in A_0 \right\}$. Since A_0 is lower finite it follows that ΣA is also

lower finite. Therefore $\Sigma A - (2\mathbb{N}) = \Gamma$ is lower finite. \square

Lemma A.2 *For any $\beta \leq 0$ and for any initial condition $w_0 \in S^\beta(\mathbb{R})$ there exists an asymptotic solution $f(x, t) \in S^\beta(\mathbb{R} \times \mathbb{R})$ of the initial value problem (1.1). Moreover, if $w_0 \sim \sum_{k=0}^{\infty} (a_k^\pm + ib_k^\pm) x^{\beta_k}$ and j is the smallest index such that $a_j^+ + ib_j^+ \neq 0$ (resp. $a_j^- + ib_j^- \neq 0$) then the coefficient $a_j^+(t) + ib_j^+(t)$ (resp. $a_j^-(t) + ib_j^-(t)$) in the asymptotic expansion of the solution is a nonvanishing continuous function of t and all preceding coefficients are identically zero.*

Proof of Lemma A.2 First we will show how to construct a formal solution $\sum_{k=0}^{\infty} (a_k^\pm + ib_k^\pm) x^{\beta_k}$.

By symmetry it suffices to construct only the positive x formal solution. For simplicity we shall omit the superscript + sign in the coefficients $a_j(t)$.

First we enlarge the exponent set $A_0 = \{\beta_j\}_{j=0}^{\infty}$ to the set Γ as defined above. From lemma A.1 it follows that we may write the set Γ as a decreasing sequence $\Gamma = \{\gamma_j\}_{j=0}^{\infty}$ where $0 \geq \gamma_0$, $\gamma_j > \gamma_{j+1}$, and $\gamma_j \rightarrow -\infty$ as $j \rightarrow \infty$, and we may rewrite the positive x asymptotic expansion of w_0 as $\sum_{j=0}^{\infty} (a_j + ib_j) x^{\gamma_j}$ where $a_j + ib_j = 0$ if $\gamma_j \notin A_0$. In order to construct the formal solution we need to solve for the coefficients $a_j(t) + ib_j(t)$ of x^{γ_j} . If $\sum_{j=0}^{\infty} (a_j(t) + ib_j(t)) x^{\gamma_j}$ is the positive x formal solution to (1.1) then,

$$\begin{aligned} & \sum_{j=0}^{\infty} [i\dot{a}_j(t) - \dot{b}_j(t)] x^{\gamma_j} \\ &= -\mu \left(\sum_{j=0}^{\infty} \left(\sum_{\substack{l,m,n \\ \gamma_l+\gamma_m+\gamma_n=\gamma_j}} [a_l(t) + ib_l(t)] \cdot [a_m(t) + ib_m(t)] \cdot [a_n(t) - ib_n(t)] \right) x^{\gamma_j} \right) \\ & \quad - \left(\sum_{j=0}^{\infty} \gamma_j \cdot (\gamma_j - 1) \cdot [a_j(t) + ib_j(t)] \cdot x^{\gamma_j-2} \right) \end{aligned}$$

from which we deduce that the coefficients $a_j(t), b_j(t)$ satisfy the equations,

$$\dot{a}_j = -\mu \sum_{\substack{l,m,n \\ \gamma_l+\gamma_m+\gamma_n=\gamma_j}} -a_l \cdot a_m \cdot b_n + a_l \cdot a_n \cdot b_m + a_n \cdot a_m \cdot b_l + b_l \cdot b_m \cdot b_n - \sum_{\substack{p \\ \gamma_p-2=\gamma_j}} b_p \cdot \gamma_p \cdot (\gamma_p - 1) \quad (\text{A.1})$$

$$\dot{b}_j = \mu \sum_{\substack{l,m,n \\ \gamma_l+\gamma_m+\gamma_n=\gamma_j}} a_l \cdot a_m \cdot a_n + a_l \cdot b_n \cdot b_m + b_n \cdot a_m \cdot b_l - b_l \cdot b_m \cdot a_n + \sum_{\substack{p \\ \gamma_p-2=\gamma_j}} a_p \cdot \gamma_p \cdot (\gamma_p - 1) \quad (\text{A.2})$$

First we will consider the case when $\gamma_0 < 0$. Notice first that for $j = 0$ the second sums of (A.1) and (A.2) are nonexistent since $\gamma_0 \geq \gamma_p$ for all $p \geq 0$ and hence there is no $p \geq 0$ such that $\gamma_p - 2 = \gamma_0$. Also for $j = 0$ the first sums are both nonexistent as well because $\gamma_l + \gamma_m + \gamma_n \leq 3\gamma_0$

and if $\gamma_0 = \gamma_l + \gamma_m + \gamma_n$ then $\gamma_0 \leq 3\gamma_0$ and hence $\gamma_0 \geq 0$ which is a contradiction to our assumption that $\gamma_0 < 0$. Thus we have $\dot{a}_0 = 0$ and $\dot{b}_0 = 0$ and hence $a_0(t) = a_0$ is constant and $b_0(t) = b_0$ is constant. Moreover, for $j \geq 0$ we can see that both sums on the right sides of (A.1) and (A.2) only contain indices less than j . To see this let us first consider the second sums. If $\gamma_p - 2 = \gamma_j$ then $\gamma_p = \gamma_j + 3 > \gamma_j$ and hence $p < j$. For the first sums, if $\gamma_j = \gamma_l + \gamma_m + \gamma_n$ and $l \geq j$ then $0 \leq \gamma_j - \gamma_l = \gamma_m + \gamma_n$ so that $\gamma_m + \gamma_n \geq 0$, but $\gamma_m, \gamma_n < 0$, so this is a contradiction, thus $l < j$. The same argument shows that $m < j$ and $n < j$. Therefore we may solve for a_j and b_j recursively by integrating the right sides of equations (A.1) and (A.2) to obtain a polynomials in t . By construction the polynomials will be identically zero for the first few indices until we reach $a_j + ib_j \neq 0$, then it will be a constant $a_j(t) + ib_j(t) = a_j + ib_j$, and for all larger indices $a_j(t) + ib_j(t)$ is polynomial and hence each $a_j(t) + ib_j(t)$ is defined for all $t \in \mathbb{R}$.

Now let us assume that $\gamma_0 = 0$. When $j = 0$ the second sums of (A.1) and (A.2) are again nonexistent for the same reason given above however the first sums are nonzero. If $\gamma_l + \gamma_m + \gamma_n = \gamma_0 = 0$ then $\gamma_l = \gamma_m = \gamma_n = 0$. Therefore $a_0(t)$ and $b_0(t)$ satisfy the equations

$$\begin{aligned}\dot{a}_0 &= -\mu(a_0^2 b_0 + b_0^3) \\ \dot{b}_0 &= \mu(a_0^3 + a_0 b_0^2)\end{aligned}$$

which can easily be solved to yield the solutions

$$\begin{aligned}a_0(t) &= a_0(0) \cos(c\mu t) + b_0(0) \sin(c\mu t) \\ b_0(t) &= -a_0(0) \sin(c\mu t) + b_0(0) \cos(c\mu t)\end{aligned}$$

where $c = a_0(0)^2 + b_0(0)^2$ and hence $a_0(t)$ and $b_0(t)$ are continuous and nonvanishing for $t \in \mathbb{R}$.

Now we'll consider the construction of functions $a_j(t), b_j(t)$ for $j \geq 1$. When $j \geq 1$ the second sums of (A.1) and (A.2) always consist of indices less than j however for the first sums all terms involve indices which are at most j . To see this, suppose that $\gamma_j = \gamma_l + \gamma_m + \gamma_n$ and suppose that $l > j$. Then $0 < \gamma_j - \gamma_l = \gamma_m + \gamma_n$ and hence $\gamma_m + \gamma_n > 0$ which is a contradiction since $\gamma_m, \gamma_n \leq 0$. Therefore we conclude that if $\gamma_j = \gamma_l + \gamma_m + \gamma_n$ and $j \geq 1$ then $l, m, n \leq j$.

Furthermore, if $l = j$ then $\gamma_m + \gamma_n = 0$ which implies that $m = n = 0$. Similar statements hold when $m = j$ and when $n = j$. Therefore when $j \geq 1$ the coefficients $a_j(t), b_j(t)$ satisfies the

equations

$$\begin{aligned}\dot{a}_j &= -\mu (2a_0 b_0 a_j + a_0^2 b_j + 3b_0^2 b_j) + P_j(a_0, b_0 \dots, a_{j-1}, b_{j-1}) \\ \dot{b}_j &= \mu (3a_j a_0^2 + a_j b_0^2 + 2a_0 b_0 b_j) + Q_j(a_0, b_0 \dots, a_{j-1}, b_{j-1})\end{aligned}$$

where P_j and Q_j are polynomials and hence the functions $a_j(t)$ and $b_j(t)$ exist and are continuous for $t \in \mathbb{R}$. This concludes the construction of the formal solution $\sum_{j=0}^{\infty} [a_j^{\pm}(t) + ib_j^{\pm}(t)] x^{\gamma_j}$.

Now let $f(x, t)$ denote any smooth function which is asymptotic to the formal solution $\sum_{j=0}^{\infty} [a_j^{\pm}(t) + ib_j^{\pm}(t)] x^{\gamma_j}$ (by proposition 3.5 in [17] there exists such a function). By plugging in $f(x, t)$ to (1.1) we will now show that one obtains a function $g(x, t) \in S^{-\infty}(\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C})$. Suppose $|x| \geq 1$ and $J \subset \mathbb{R}$ is a compact subset and $l, m, N \geq 0$ are integers. Let $S_N(f) = \sum_{j=0}^N [a_j^{\pm}(t) + ib_j^{\pm}(t)] x^{\gamma_j}$ (i.e. if $x > 1$ then $S_N(f) = \sum_{j=0}^N [a_j^+(t) + ib_j^+(t)] (+x)^{\gamma_j}$ and when $x < -1$ we have $S_N(f) = \sum_{j=0}^N [a_j^-(t) + ib_j^-(t)] (-x)^{\gamma_j}$). Then we have

$$\begin{aligned}\partial_t^l \partial_x^m (if_t + f_{xx} + \mu \bar{f} f^2) &= i \partial_t^l \partial_x^m [S_N(f) + (f - S_N(f))]_t + \partial_t^l \partial_x^m [S_N(f) + (f - S_N(f))]_{xx} \\ &\quad + \mu \partial_t^l \partial_x^m \left[\overline{(S_N(f) + (f - S_N(f)))} (S_N(f) + (f - S_N(f)))^2 \right]\end{aligned}$$

After expanding the right side we will obtain the expression $\partial_t^l \partial_x^m \left[iS_N(f)_t + S_N(f)_{xx} + \mu \overline{S_N(f)} S_N(f)^2 \right]$ and terms which are products constants times $\partial_t^p \partial_x^q (f - S_N(f))^r$ and $\partial_t^s \partial_x^k S_N(f)^n$ for some $p, q, r, s, n \in \mathbb{N}$. Since the coefficients $a_j(t), b_j(t)$ satisfy the equations (A.1) and (A.2) and since f is asymptotic to $\sum_{j=0}^{\infty} [a_j^{\pm}(t) + ib_j^{\pm}(t)] x^{\gamma_j}$ we have for any $M \in \mathbb{N}$ there exists a sufficiently large N such that these terms are bounded by $C_{M, J, l, m} |x|^{-M}$. \square

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